# Transformation and seasonal adjustment

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**Abstract**

This paper considers the effects of seasonal adjustment on transformed time series which are then transformed back to provide seasonally adjusted series in the original scale of the observations. It is shown that this approach can lead to trend and seasonal biases, particularly where there is significant variation about the trend due to either or both of the seasonal or irregular components. Bias correction methods are given and results are illustrated by simulation and with reference to New Zealand official time series.

**Keywords:** Seasonal adjustment; Trend estimation; Transformation; Bias correction.

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# 1 Introduction

Many time series, particularly monthly economic and official time series, are both non-linear and seasonal. Seasonal adjustment is concerned with the identification and removal of seasonal variation from such time series. A main objective is to provide data free of seasonal variation which might otherwise confound simple estimates of trend. Indeed, the concept of trend is central to good seasonal adjustment and the accurate identification of trends from seasonally adjusted data is important if only to determine important trend parameters such as direction, level and rate of change, or for the purpose of comparison between series. For the sake of definiteness we restrict attention to monthly time series with annual seasonality although our results and observations apply more generally.

A broad class of non-linear seasonal time series models widely used in practice is given by the additive model

$$\phi(Y_t) = T_t + S_t + \epsilon_t \tag{1}$$

where $Y_t$ denotes the original series, $\phi(y)$ a suitable transformation, $T_t$ and $S_t$ the trend and seasonal components and the so-called irregular component $\epsilon_t$ denotes noise. Other components such as a calendar component can also be added. To further identify the components in (1) additional constraints are needed. These include smoothness constraints for the trend and the year to year evolution of the seasonal together with the requirement that the seasonal component sums approximately to zero over any twelve month period so that

$$\sum_{j=0}^{11} S_{t-j} \approx 0. \tag{2}$$

In addition the noise is assumed to be stationary with mean zero (often white noise), and all three components in the additive decomposition are assumed to be independent. These constraints are typcally chosen so that $T_t$ is locally smooth and runs through the middle of the transformed data over any twelve month period.

The most widespread transformations used are the identity $\phi(y) = y$ for the simple additive model and the logarithm $\phi(y) = \log y$ for data whose components are multiplicative. The seasonal adjustment procedure SABL (Cleveland et al (1978)) augments these by considering the class of power transformations defined by

$$\phi(y) = \begin{cases} y^p & (p > 0) \\ \log y & (p = 0) \\ -y^p & (p < 0) \end{cases} . \tag{3}$$

The purpose of this transformation is to make all three components in (1) as independent as possible. In practice $\phi(y)$ is chosen so that there is no interaction between the trend and seasonal components. However the seasonal adjustment procedures used by most official statistical agencies in the world are X-11-ARIMA (Dagum (1980)) and X-12-ARIMA (Findley et al (1988)). These are based on the original X-11 procedure (Shiskin et al (1967)) which has its own multiplicative model which is different from (1) with $\phi(y) = \log y$. A useful reference to seasonal models such as (1) and seasonal adjustment procedures in general is given in the survey article Cleveland (1983).

Consider the case where $Y_t$ follows (1) with $\phi(y)$ not the identity transformation. To seasonally adjust $Y_t$ it is common practice to seasonally adjust the transformed series $\phi(Y_t)$ by removing the seasonal component and then transforming back into the original scale of the observations. Seasonally adjusted trends can also be obtained by back-transforming the trend of the transformed series. However this approach can lead to ambiguities in terms of definition of trend and seasonal, particularly where there is significant variation about the trend, due to either or both of the seasonal or irregular components.

An example is given in Figure 1 which shows the number of visitor arrivals to New Zealand by month over the period January 1980 to December 1991. Three trends are superimposed: the trend estimated by the X-11 multiplicative model and two SABL trends, one of the untransformed data and the other, the exponential of the trend of the logarithms of the data. Despite the fact that X-11 trends are locally quadratic and SABL trends are locally linear, both the X-11 and SABL trends of the original data are much the same. However there is a significant difference between these and the exponential of the trend of the logarithms of visitor arrivals. This difference, essentially the difference between the arithmetic and geometric means, was first systematically discussed in the literature by Young (1968). His correction formulae for the multiplicative model are closely related to the more general formulae proposed here. beginfigure

The reason why the X-11 multiplicative trend yields much the same trend as the SABL trend from the original visitor data is discussed in Section 2. However the important point to note is the following. If an additive model is fitted to the transformed data using (1) and then the inverse transform applied, the trend, seasonal and seasonally adjusted series obtained are not independent of the transformation chosen. Systematic differences exist between them. In many cases the differences are slight. However in cases where there is significant variability about the trend due to either or both of the seasonal or irregular, the difference can be sizeable. Indeed, for the New Zealand Visitor Arrivals data given in Figure 1, the difference between the two SABL trends during June 1988 is 1839 arrivals or 2.5%.

A situation where such differences are important concerns balance of payments series comprising credits less debits where the two components (credits and debits) each follow multiplicative models with strong trends and stable seasonals, but the balance has a weak trend and unstable seasonal. If one chooses to seasonally adjust the balance series by taking the difference of the (stable) seasonally adjusted credits and debits series then spurious systematic trend differences may occur. If the original balance series is seasonally adjusted directly without taking account of the structure of its multiplicative components then this typically leads to unstable seasonal patterns and, as a consequence, unstable seasonal adjustments result. These issues are further discussed in Section 2.

The above discussion highlights the importance of defining appropriate trend and seasonal components in the case where $\phi(y)$ is not the identity transformation. The standard requirement that is built in to both the additive and multiplicative X-11 models is that (moving) annual totals of seasonally adjusted and unadjusted series should be essentially the same. We shall refer to this as the *seasonal balance constraint.* This is an economic requirement which ensures that the process of seasonal adjustment is essentially one of

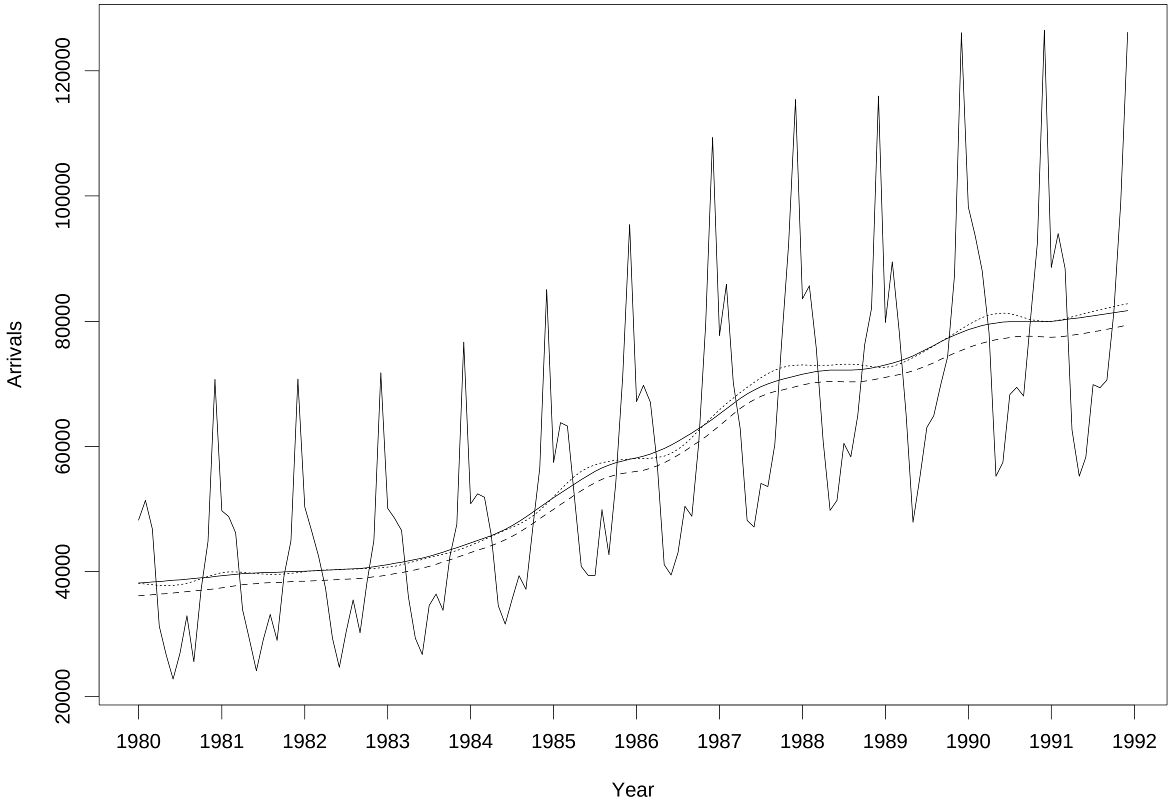



Figure 1: Visitor arrivals to New Zealand by month. Three trends are superimposed: the trend estimated by the X-11 multiplicative model (dotted line) and two SABL trends, one of the untransformed data (solid line) and the other the exponential of the trend of the logarithms of the data (dashed line)

redistribution of seasonal variation so that, on an annual basis, observed totals (wealth, credits, debits, numbers of visitor arrivals etc) are neither created nor destroyed. Although this is a natural economic requirement, for transformed series following (1) it conflicts with (2) except in the additive case $\phi(y) = y$. In Section 3 we use the seasonal balance constraint to define appropriate trend and seasonal components for the original series $Y_t$.

## 2 X-11 Models

Before developing appropriate definitions of trend, seasonal and irregular for models such as (1) where $\phi(y) \neq y$, it is instructive to review the X-11 multiplicative model.

X-11 was initially developed by Shiskin et al (1967). Despite many new developments in seasonal time series models, it remains the most popular method of seasonal adjustment and forms the basis of X-11-ARIMA (Dagum (1980)) and X-12-ARIMA (Findley et al (1988)) used by the majority of the world's official statistical agencies. The additive version of X-11 is as given in (1) with $\phi(y) = y$. However the multiplicative version of

X-11 is essentially based on the model

$$Y_t = T_t(1 + S_t)(1 + \epsilon_t) \tag{4}$$

where the seasonal factor $S_t$ satisfies (2) and $E(\epsilon_t) = 0$. Applying the logarithm transformation to (4) yields a model which is approximately the same as (1) with $\phi(y) = \log y$ provided that $S_t$ and $\epsilon_t$ are small. If either or both of the seasonal or irregular are sizeable then differences emerge as is evidenced in Fig. 1. In particular the two seasonal constraints differ with one requiring the moving annual arithmetic mean, the other the moving annual geometric mean, of the seasonals to be approximately zero.

The model (4) can also be written in additive form as

$$Y_t = T_t + T_tS_t + T_t(1 + S_t)\epsilon_t. \tag{5}$$

Provided that the trend is smooth, (2) ensures that the seasonal component $T_tS_t$ will approximately sum to zero over any twelve month period. This model is similar to that proposed by Durbin and Murphy (1975), but has heteroskedastic multiplicative errors rather than additive homoskedastic errors.

The additive form (5) helps to explain the non-linear X-11 model fitting procedure. Broadly speaking, the trend $T_t$ is first estimated by filtering (5) with a linear low-pass filter that also removes the seasonal $T_tS_t$. The resulting trend estimate is then divided through (5) and the seasonal component $1 + S_t$ extracted using simple linear filters. This is then divided into (4) to obtain a seasonally adjusted series. Further smoothing iterations are carried out together with various procedures to down-weight outliers, adjust for calendar effects etc. Thus it can be argued that the X-11 procedure expresses a multiplicative model as an additive model and fits accordingly. This has the advantage of ensuring that the seasonal balance constraint is satisfied and is the reason why, in Figure 1, the X-11 multiplicative trend of the NZ visitor arrivals series is much the same as the SABL trend from the untransformed data.

Using the X-11 model (4) the comments made in Section 1 concerning balance of payments series can now be further amplified. Let us suppose that the balance of payments series is given by $B_t = C_t - D_t$ where credits $C_t$ and debits $D_t$ follow the multiplicative X-11 models

$$C_t = T_t^C(1 + S_t^C)(1 + \epsilon_t^C) \qquad D_t = T_t^D(1 + S_t^D)(1 + \epsilon_t^D) \tag{6}$$

and $S_t^C$,$S_t^D$ satisfy (2). In general, no simple transformation will reduce $B_t$ to a simple additive model whose trend and seasonal are stable with no interaction. Moreover, since $B_t$ will typically take on positive and negative values, the power transformation chosen by a package such as SABL will almost always be the simple identity transformation.

However, from (5), $B_t$ does admit the additive decomposition $B_t = T_t^B + S_t^B + \epsilon_t^B$ where

$$\begin{aligned} T_t^B &= T_t^C - T_t^D \\ S_t^B &= T_t^C(S_t^C - S_t^D) + (T_t^C - T_t^D)S_t^D \\ \epsilon_t^B &= T_t^C(1 + S_t^C)\epsilon_t^C - T_t^D(1 + S_t^D)\epsilon_t^D \end{aligned} \tag{7}$$

The difficulty of seasonally adjusting $B_t$ directly is evident. Suppose $T_t^C$ and $T_t^D$ are strong and evolving slowly with a proportional difference that is stable and near zero. Then $B_t$ will have a trend $T_t^B$ that evolves smoothly, but with sizeable excursions about zero, and the seasonal pattern will be unstable with changing amplitude and shape. Indeed, if $T_t^C \approx T_t^D$ and $S_t^C \approx S_t^D$ then $S_t^B \approx 0$ whereas if $T_t^C \approx T_t^D$ and $S_t^C \approx -S_t^D$ then $S_t^B \approx 2T_t^C S_t^C$. With evolving seasonal patterns the truth will lie somewhere between these two extremes resulting in unstable and transient seasonal patterns for $B_t$. In this particular case one should use an X-11 based procedure and seasonally adjust the component series $C_t$ and $D_t$ rather than the balance series $B_t$. However, if this is done using other seasonal adjustment procedures, then care must be taken to correct for any biases in estimated trend and seasonal components that might result.

The balance of payments example is a particular case of the general aggregated non-linear model

$$\phi(Y_t) = \sum_{j=1}^{p} \alpha_j Y_t^{(j)} \tag{8}$$

where the $Y_t^{(j)}$ follow (1) with

$$\phi_j(Y_t^{(j)}) = T_t^{(j)} + S_t^{(j)} + \epsilon_t^{(j)} \qquad (j = 1 \cdots p)$$

and $\phi(y)$, $\phi_j(y)$ denote appropriate power transformations. This not only encompasses models such as (1) and the balance of payments example, but also covers many other situations. Consider, for example, the ratio of unemployment and labour force where numerator and denomenator follow multiplicative models. Here $Y_t$ denotes the required ratio, $Y_t^{(1)}$, $Y_t^{(2)}$ denote the logarithms of unemployment and labour force respectively, $\phi(y) = \log y$, $\phi_1(y) = \phi_2(y) = y$ and $\alpha_1 = -\alpha_2 = 1$.

It is generally regarded that disaggregation of a time series prior to seasonal adjustment is advisable (see Geweke (1976)). The balance of payments example illustrated above is an obvious example where this strategy has much to commend it. However, if the individual component series are individually seasonally adjusted, then any biases in the estimated trends will clearly affect estimates of trend and seasonal. These biases may well have unwanted and undesirable effects. This example highlights the need to define appropriate trend and seasonal components for non-linear seasonal time series such as (1) and (8).

## 3 Bias correction formulae

The general strategy we adopt is to express $Y_t$ in additive form with trend, seasonal and irregular defined as functions of the corresponding components in the transformed series. The derived trend, seasonal and seasonally adjusted series are then constructed from the estimates of the components in the transformed series. This approach is implicit in the X-11 multiplicative procedure. It has the virtue of separating the model fitting, which takes place with the transformed series, from the additive trend-seasonal decomposition of the original series $Y_t$ which is constructed so that the seasonal balance constraint remains approximately true.

Before constructing an additive decomposition for $Y_t$, we first need to make some basic assumptons about the trend, seasonal and irregular components given in (1). We shall assume that all three components are independent and that the trend $T_t$ follows a deterministic or stochastic model which is locally smooth. For example X-11 and SABL assume that $T_t$ follows a local low-order deterministic polynomial model in $t$ within a moving window of consecutive monthly observations. On the other hand Akaike (1980), Gersch and Kitagawa (1983), Harvey (1989) and many others assume global stochastic models for $T_t$ of the form

$$\Delta^p T_t = \eta_t$$

where $\eta_t$ is white noise and $\Delta$ is the backwards difference operator satisfying $\Delta X_t = X_t - X_{t-1}$. Typically $p = 1$ or 2 and the variance of $\eta_t$ is taken small enough to ensure that $T_t$ is smooth. The irregular component $\epsilon_t$ is assumed to be a zero-mean stationary time series.

We shall assume that the evolving seasonal component can always be represented as

$$S_t = \sum_{j=1}^{6} \{\alpha_j(t) \cos t\lambda_j + \beta_j(t) \sin t\lambda_j\} \tag{9}$$

where $\lambda_j = 2\pi j/12$ and $\beta_6(t) = 0$. The zero frequency component corresponding to $\lambda_0 = 0$ has been omitted from (9) to ensure that $S_t$ indeed measures departures from the trend $T_t$ which describes the instantaneous local level of $\phi(Y_t)$ at time $t$. As a consequence this representation automatically satisfies (2) since $\sum_{j=0}^{11} S_{t-j}$ will be approximately zero provided that the $\alpha_j(t)$ and $\beta_j(t)$ are sufficiently smooth. Note that it will rarely be the case that $\sum_{j=0}^{11} S_{t-j}$ is identically zero. This will only be true when $S_t$ is strictly periodic and the $\alpha_j(t)$, $\beta_j(t)$ are constants. If, as is commonly the case, the $\alpha_j(t)$, $\beta_j(t)$ are evolving slowly over time then $S_t$ will also evolve slowly and $\sum_{j=0}^{11} S_{t-j}$ will have the appearance of a stationary time series with small variance. However, in the case where the $\alpha_j(t)$, $\beta_j(t)$ are evolving linearly in $t$, then $\sum_{j=0}^{11} S_{t-j}$ may also exhibit a seasonal pattern which, in turn, will need to approximately sum to zero over any twelve month period. In this case (2) will need to be replaced by $\sum_{j=0}^{11} \sum_{k=0}^{11} S_{t-j-k} \approx 0$.

The model (9) is implicitly used by X-11 and SABL where the $\alpha_j(t)$, $\beta_j(t)$ are assumed to be constant or linear in $t$ within an appropriately defined moving window. The mixed model proposed by Durbin and Murphy (1975) also fits this framework. In terms of stochastic seasonal models, (9) is the same as the model proposed by Hannan (1967) (see also Ng and Young (1990)) where $\alpha_j(t)$, $\beta_j(t)$ follow the stochastic trend models

$$\Delta\alpha_j(t) = \eta_j(t), \qquad \Delta\beta_j(t) = \xi_j(t) \tag{10}$$

and the white noise processes $\eta_j(t)$ $(j = 1, \ldots, 6)$ and $\xi_j(t)$ $(j = 1, \ldots, 5)$ are mutually independent with $E(\eta_j(t)^2) = E(\xi_j(t)^2) = \sigma_j^2$. In fact this stochastic seasonal model is more general than might first appear. Dongfeng et al (1997) show that Hannan's model is stochastically equivalent to the model proposed by Harvey (1989) where

$$S_t = \sum_{j=1}^{6} u_j(t)$$

and

$$\begin{pmatrix} u_j(t) \\ v_j(t) \end{pmatrix} = \begin{pmatrix} \cos\lambda_j & \sin\lambda_j \\ -\sin\lambda_j & \cos\lambda_j \end{pmatrix} \begin{pmatrix} u_j(t-1) \\ v_j(t-1) \end{pmatrix} + \begin{pmatrix} \eta_j(t) \\ \xi_j(t) \end{pmatrix}$$

with the $\eta_j(t)$, $\xi_j(t)$ defined as in (10). This seasonal model is used in the structural time series modelling procedure STAMP (Koopman et al (1995) and the seasonal decomposition procedure MING (Bruce and Jurke (1996)). Moreover, if the $\eta_j(t)$, $\xi_j(t)$ are replaced by carefully chosen stationary processes, then (9) can be shown to encompass the stochastic seasonal model

$$\sum_{j=0}^{11} S_{t-j} = \eta_t$$

where $\eta_t$ is stationary. The case where $\eta_t$ is white noise is commonly used (see Gersch and Kitagawa (1983), Harvey (1989) for example). If $\eta_t$ is an AR(1) process then the seasonal model is equivalent to that used in BAYSEA (Akaike (1980)). The generality of (9) and the properties of such stochastic seasonal models is the subject of ongoing research with some results already reported in Dongfeng et al (1997). It is sufficient for our purposes here that we assume that $S_t$ admits the instantaneous Fourier representation (9) with the $\alpha_j(t)$, $\beta_j(t)$ evolving slowly over time.

To handle both (1) and (4), we now consider

$$Y_t = g(T_t, S_t, \epsilon_t) \tag{11}$$

where the components $T_t$, $S_t$, $\epsilon_t$ are as defined above and $g(x,s,e)$ is either $\phi^{-1}(x+s+e)$ with $\phi(y)$ given by (3) or the X-11 model $x(1+s)(1+e)$ given by (4). More generally $g(x,s,e)$ could be any well-behaved function of its arguments. We write $Y_t$ in the additive form

$$Y_t = T_t^* + S_t^* + \epsilon_t^* \tag{12}$$

where $T_t^*$, $S_t^*$, $\epsilon_t^*$ are yet to be defined trend, seasonal and irregular components. Proceeding constructively we define $M_t = T_t^* + S_t^*$ as

$$M_t = E\{g(T_t, S_t, \epsilon_t)|\mathbf{T}, \mathbf{S}\} \tag{13}$$

where $\mathbf{T}$ and $\mathbf{S}$ denote the processes $\{T_t; t = 0, \pm 1, \ldots\}$ and $\{S_t; t = 0, \pm 1, \ldots\}$ respectively. This additively decomposes $Y_t$ into a systematic component $M_t$ and an irregular component

$$\epsilon_t^* = Y_t - M_t \tag{14}$$

where $\epsilon_t^*$ has zero mean and is uncorrelated with $M_t$. Moreover $M_t$ is an evolving trend and seasonal pattern that is a function $M(x,s)$ of $T_t$ and $S_t$ which, from (9), can be written as

$$M_t = M(T_t, S_t) = M(T_t, \sum_{j=1}^{6}\{\alpha_j(t)\cos t\lambda_j + \beta_j(t)\sin t\lambda_j\}).$$

Now any function of an (instantaneous) Fourier representation will create a new (additive) Fourier representation of the form (9) with new coefficients $\alpha_j(t)$, $\beta_j(t)$ and, in particular, an additional zero frequency ($\lambda_0 = 0$) or local level component with $\beta_0(t) = 0$, but $\alpha_0(t)$ not necessarily zero. For example, if $M(x, s) = (x + s)^2$ then

$$M(T_t, S_t) = (\sum_{j=0}^{6}\{\alpha_j(t)\cos t\lambda_j + \beta_j(t)\sin t\lambda_j\})^2 = \sum_{j=0}^{6}\{a_j(t)\cos t\lambda_j + b_j(t)\sin t\lambda_j\}$$

where $\alpha_0(t) = T_t$, $\beta_0(t) = 0$ and

$$a_0(t) = \alpha_0(t)^2 + \sum_{j=1}^{5}\{\alpha_j(t)^2 + \beta_j(t)^2\} + \alpha_6(t)^2, \qquad b_0(t) = 0.$$

This can be verified by direct algebra or by using the formula for the Fourier coefficients which, for $\lambda_0 = 0$ gives

$$a_0(t) = \frac{1}{12}\sum_{k=0}^{11}(\sum_{j=0}^{6}\{\alpha_j(t)\cos(t-k)\lambda_j + \beta_j(t)\sin(t-k)\lambda_j\})^2.$$

These arguments lead us us to define the instantaneous local level of $Y_t$ at time $t$ as

$$T_t^* = E\{\frac{1}{12}\sum_{k=0}^{11} M(T_t, \sum_{j=1}^{6}\{\alpha_j(t)\cos(t-k)\lambda_j + \beta_j(t)\sin(t-k)\lambda_j\})|\mathbf{T}, \mathbf{S}\} \tag{15}$$

where the conditional expectation ensures that $T_t^*$ is a function of $T_t$ and $\mathbf{S}$. However the conditional expectation will not be necessary in the special case where the $\alpha_j(t)$, $\beta_j(t)$ are extractly directly from the transformed series. Note that the second argument of $M$ is not $S_{t-k}$ unless the $\alpha_j(t)$, $\beta_j(t)$ are locally constants. The seasonal component $S_t^*$ is now defined by subtraction as

$$S_t^* = M_t - T_t^*. \tag{16}$$

In particular, $T_t^* = T_t$, $S_t^* = S_t$ and $\epsilon_t^* = \epsilon_t$ in the case of the identity transformation where $g(x, s, e) = x + s + e$, and $T_t^* = T_t$, $S_t^* = T_t S_t$ and $\epsilon_t^* = T_t(1 + S_t)\epsilon_t$ in the X-11 case where $g(x, s, e) = x(1 + s)(1 + e)$. This leads to the following definition.

**Definition 1** *Let $Y_t$ follow the model specified by (11). Then the trend $T_t^*$, the seasonal $S_t^*$ and the irregular $\epsilon_t^*$ in the additive decomposition (12) of $Y_t$ are defined by (13), (14), (15) and (16). In particular, the identity transformation model (1) with $p = 0$ and the X-11 model (5) have this additive form.*

Given any particular model for $Y_t$ of the form (11), one could derive precise formulae for $T_t^*$, $S_t^*$ and $\epsilon_t^*$ as functions of $T_t$, $S_t$ and $\epsilon_t$. Then these functions and the estimated trend, seasonal and irregular of the transformed series could be used to provide estimates of the trend, seasonal and irregular for the original series. Although of interest, this approach has not been followed here. Rather we shall adopt a simpler strategy of approximating these functions with simple non-parametric linear filters of functions of the component series. For any given function $h(x)$, it is sufficient to consider approximating

$$E\{\frac{1}{12}\sum_{k=0}^{11} h(\sum_{j=1}^{6}\{\alpha_j(t)\cos(t-k)\lambda_j + \beta_j(t)\sin(t-k)\lambda_j\})|\mathbf{T}, \mathbf{S}\} \tag{17}$$

by some suitable linear filter of the process $h(S_t)$. Denote this approximating filter by $L_S$ where

$$L_S(h(S_t)) = \sum c_k h(S_{t-k}). \tag{18}$$

If the $\alpha_j(t)$, $\beta_j(t)$ are locally constant then (17) becomes

$$\frac{1}{12}\sum_{k=0}^{11} h(S_{t-k})$$

and in this case $L_S$ is just the simple one-sided 12 month moving average. However, to allow for evolution in the $\alpha_j(t)$, $\beta_j(t)$, other linear filters based on the simple 12 month moving average might better be employed in practice. These include the standard 12 month (13 point) centred moving average where the non zero $c_j$ are given by

$$c_{-6} = \frac{1}{24}, \quad c_{-5} = c_{-4} = \ldots = c_4 = c_5 = \frac{1}{12}, \quad c_6 = \frac{1}{24}$$

or the triangular 23 point moving average where the non zero $c_k$ are given by

$$c_k = \frac{12 - |k|}{144} \qquad (k = 0, \pm 1, \ldots, \pm 11).$$

In particular, the latter filter results if the $\alpha_j(t)$, $\beta_j(t)$ are locally linear. Then it is not hard to show that (17) becomes

$$\frac{1}{12}\sum_{k=0}^{11} h((1 - \frac{k}{12})S_{t-k} + \frac{k}{12}S_{t+12-k})$$

which, using linear interpolation, should be well approximated by

$$\frac{1}{12}\sum_{k=0}^{11}((1 - \frac{k}{12})h(S_{t-k}) + \frac{k}{12}h(S_{t+12-k})) = \sum_{k=-11}^{11}\frac{12 - |k|}{144}h(S_{t-k}).$$

The quality of this approximation is dependent on the smoothness of $h(x)$ and the closeness of the values of $S_t$ for the same month in successive years. However it might be expected that these would be quite reasonable approximations in practice. Finally we note that (17) is identically zero when $h(x) = x$, regardless of the law of evolution for the $\alpha_j(t)$, $\beta_j(t)$. This leads to the convention that $L_S(S_t)$ is set to zero. These approximations and definitions are now used to define suitable non-parametric approximations of $T_t^*$ and $S_t^*$.

Assuming that $M(x, s)$ can be expanded in its second argument as a Taylor series expansion about 0, we are lead to approximate $T_t^*$ and $S_t^*$ by

$$\begin{aligned} \hat{T}_t^* &= M(T_t, 0) + \sum_{j=2}^{\infty} m_j(T_t) L_S(S_t^j)/j! \\ \hat{S}_t^* &= M_t - \hat{T}_t^* \end{aligned} \tag{19}$$

where $M_t$ is given by (13) and $m_j(x)$ denotes the $j$th partial derivative of $M(x, s)$ with respect to $s$ evaluated at $s = 0$. This leads to the following result.

**Result 2** *Let $Y_t$ follow the model specified by (11) with additive decomposition given by Definition 1. Then the expressions (19) and (14) provide an approximate non-parametric additive decomposition of $Y_t$ in cases other than the identity transformation model (1) with $p = 0$ and the X-11 model (5).*

Consider, for example, (1) and the cases $p = 0$ and $p = 0.5$. For the multiplicative case $\phi(y) = \log y$ and

$$M_t = \psi(1)e^{T_t+S_t}, \qquad \hat{T}_t^* = \psi(1)e^{T_t}L_S(e^{S_t}), \qquad \hat{S}_t^* = \psi(1)e^{T_t}(e^{S_t} - L_S(e^{S_t}))$$

with $\psi(s)$ denoting the moment generating function of $\epsilon_t$. In the case of Gaussian errors $\psi(1) = \exp \frac{1}{2}\sigma^2$ where $\sigma^2 = E\epsilon_t^2$. In the case of the square-root transform ($p = 0.5$) we have

$$M_t = (T_t + S_t)^2 + \sigma^2, \qquad \hat{T}_t^* = T_t^2 + L_S(S_t^2) + \sigma^2, \qquad \hat{S}_t^* = 2T_tS_t + S_t^2 - L_S(S_t^2).$$

The functional dependence of $\hat{T}_t^*$ and $\hat{S}_t^*$ on $T_t$ and $S_t$ can now be utilised to construct estimates. In the simplest case this means replacing $T_t$ and $S_t$ in (19) by their estimates obtained from the transformed data using standard seasonal-trend decomposition procedures. The unknown $\epsilon_t$ parameters in $M_t$ are estimated from the estimated irregular of the transformed data. Depending on the nature of the data, these particular estimates could be simple moment estimates or robust estimates that take proper account of outliers. In the multiplicative case this procedure yields formulae that are closely related to those advocated by Young (1968). For parametric Gaussian models such as those advocated by Akaike (1980), Gersch and Kitagawa (1983), Harvey (1989), Ng and Young (1990) among many others, better estimates of $M_t$ and $\hat{T}_t^*$ can be obtained directly by determining

$$E\{M_t|\text{data}\}, \qquad E\{\hat{T}_t^*|\text{data}\} \tag{20}$$

using the relevant conditional densities determined from the Kalman filter and smoother. Using this technology, confidence limits can also be constructed.

Now consider the case where $Y_t$ follows (1) and $\phi(y)$ is given by (3) or, more generally, by some other well-behaved function. In this case computationally simpler approximate procedures can be used provided $S_t$ and $\epsilon_t$ are small.

**Result 3** *Let $Y_t$ follow the model specified by (1) where $\phi(y)$ is some well-behaved function. Then $T_t^*$, $S_t^*$, $\epsilon_t^*$ of Definition 1 can be approximated by the simpler forms $\tilde{T}_t^*$, $\tilde{S}_t^*$, $\tilde{\epsilon}_t^*$ respectively where*

$$\begin{aligned} \tilde{T}_t^* &= \phi^{-1}(T_t + \tfrac{1}{2}\gamma(T_t)(L_S(S_t^2) + \sigma^2)) \\ \tilde{S}_t^* &= \phi^{-1}(T_t + S_t + \tfrac{1}{2}\gamma(T_t)\sigma^2) - \tilde{T}_t^* \\ \tilde{\epsilon}_t^* &= Y_t - \tilde{T}_t^* - \tilde{S}_t^* \end{aligned} \tag{21}$$

*provided $S_t$ and $\epsilon_t$ are small. Here*

$$\gamma(x) = -\phi^{(2)}(\phi^{-1}(x))/(\phi^{(1)}(\phi^{-1}(x)))^2$$

*and $\phi^{(j)}(y)$ denotes the $j$th derivative of $\phi(y)$. When $\phi(y)$ is given by the power transformation (3)*

$$\gamma(x) = \begin{cases} 1 & (p=0) \\ (p^{-1}-1)/x & (p \neq 0) \end{cases} . \tag{22}$$

The approximations given by Result 3 make simple additive adjustments to the trend of the transformed data and then back-transform. Alternative approximations can also be devised which make additive adjustments in the original scale of the observations yielding the following result.

**Result 4** *Let $Y_t$ follow the model specified by (1) where $\phi(y)$ is some well-behaved function. Then $T_t^*$, $S_t^*$, $\epsilon_t^*$ of Definition 1 can be approximated by the simpler forms $\bar{T}_t^*$, $\bar{S}_t^*$, $\bar{\epsilon}_t^*$ respectively where*

$$\begin{aligned} \bar{T}_t^* &= \phi^{-1}(T_t)(1 + \tfrac{1}{2}\delta(T_t)\gamma(T_t)(L_S(S_t^2) + \sigma^2)) \\ \bar{S}_t^* &= \phi^{-1}(T_t)\delta(T_t)(S_t + \tfrac{1}{2}\gamma(T_t)(S_t^2 - L_S(S_t^2))) \\ \bar{\epsilon}_t^* &= Y_t - \bar{T}_t^* - \bar{S}_t^* \end{aligned} \tag{23}$$

*provided $S_t$ and $\epsilon_t$ are small. Here*

$$\delta(x) = \frac{d}{dx}\log\phi^{-1}(x)$$

*and $\gamma(x)$ is as given in Result 3. When $\phi(y)$ is given by the power transformation (3)*

$$\delta(x) = \begin{cases} 1 & (p=0) \\ p^{-1}/x & (p \neq 0) \end{cases} . \tag{24}$$

## Proof to Results 3 and 4

We first consider Result 4 and the function

$$Q_M(\theta) = E\{\phi^{-1}(T_t + \theta(S_t + \epsilon_t))|\mathbf{T}, \mathbf{S}\}.$$

Expanding $Q_M(\theta)$ in a second-order Taylor series about zero and then setting $\theta = 1$ we obtain

$$M_t \approx \phi^{-1}(T_t) + \kappa_1(T_t)S_t + \tfrac{1}{2}\kappa_2(T_t)(S_t^2 + \sigma^2)$$

where $\sigma^2$ denotes the variance of $\epsilon_t$, the function $\kappa_j(x)$ denotes the $j$th derivative of $\phi^{-1}(x)$ and the error of approximation is $O(E\{(S_t+\epsilon_t)^3|\mathbf{T},\mathbf{S}\})$. Noting that

$$\kappa_1(x) = \phi^{-1}(x)\delta(x), \qquad \kappa_2(x) = \phi^{-1}(x)\delta(x)\gamma(x)$$

we obtain

$$M_t \approx \phi^{-1}(T_t)(1 + \delta(T_t)S_t + \tfrac{1}{2}\delta(T_t)\gamma(T_t)(S_t^2 + \sigma^2)).$$

Now consider

$$Q_T(\theta) = \frac{1}{12}\sum_{k=0}^{11} E\{\phi^{-1}(T_t + \theta(\bar{S}_{t-k} + \epsilon_t))|\mathbf{T}, \mathbf{S}\}$$

where

$$\bar{S}_{t-k} = \sum_{j=1}^{6} \{\alpha_j(t)\cos(t-k)\lambda_j + \beta_j(t)\sin(t-k)\lambda_j\}).$$

Expanding $Q_T(\theta)$ in the same way as before and noting that $\sum_{k=0}^{11} \bar{S}_{t-k} = 0$ we obtain

$$T_t^* \approx \phi^{-1}(T_t) + \tfrac{1}{2}\kappa_2(T_t)(\frac{1}{12}\sum_{k=0}^{11} E\{\bar{S}_{t-k}^2|\mathbf{T},\mathbf{S}\} + \sigma^2)$$

which, using the approximation developed for (17), becomes

$$T_t^* \approx \phi^{-1}(T_t)(1 + \tfrac{1}{2}\delta(T_t)\gamma(T_t)(L_S(S_t^2) + \sigma^2)$$

and the error of approximation is now $O(L_S(E\{(S_t + \epsilon_t)^3)|\mathbf{T},\mathbf{S}\}))$. These two approximations establish Result 4.

For Result 3 the two functions

$$\bar{Q}_M(\theta) = \phi(Q_M(\theta)), \qquad \bar{Q}_T(\theta) = \phi(Q_T(\theta))$$

are expanded in second-order Taylor series about zero in much the same way as before to obtain the stated results.

□

Note that in the case of (3) and the identity transformation $p = 1$ the components given by Results 3 and 4 are identical with those of Definition 1. In the multiplicative case where $\phi(y) = \log y$

$$\tilde{T}_t^* = e^{T_t + \frac{1}{2}(L_S(S_t^2)+\sigma^2)}, \qquad \tilde{S}_t^* = e^{T_t + \frac{1}{2}\sigma^2}(e^{S_t} - e^{\frac{1}{2}L_S(S_t^2)})$$

and

$$\bar{T}_t^* = e^{T_t}(1 + \tfrac{1}{2}(L_S(S_t^2) + \sigma^2)), \qquad \bar{S}_t^* = e^{T_t}(S_t + \tfrac{1}{2}(S_t^2 - L_S(S_t^2))).$$

If $\phi(y)$ is the square-root transform (3) with $p = 0.5$ then

$$\tilde{T}_t^* = (T_t + \tfrac{1}{2}(L_S(S_t^2) + \sigma^2)/T_t)^2, \qquad \tilde{S}_t^* = (T_t + S_t + \tfrac{1}{2}\sigma^2/T_t)^2 - (\tilde{T}_t^*)^2$$

and

$$\bar{T}_t^* = T_t^2 + L_S(S_t^2) + \sigma^2 = \hat{T}_t^*, \qquad \bar{S}_t^* = 2T_tS_t + S_t^2 - L_S(S_t^2) = \hat{S}_t^*.$$

In the next section the relative performance of these various procedures is investigated by analysis of simulated and real data.

In practice the above will need to be modified to handle calendar and holiday effects, and to incorporate robust estimation procedures to cope with outliers. The latter is directly addressed by estimating $M_t$ using the robustness weights derived when processing the transformed data. Calendar and holiday effects are typically modelled by adding in an extra fixed effects regression component to the right-hand side of (1). The regressors include month length, numbers of each type of week-day in the month and dummy variables for holidays. A similar development to that leading to Definition 1 could be undertaken to define appropriate additive components in the original scale of the observations. However, once mean corrected, these effects are typically sufficiently small that they do not influence the definitions of trend and seasonal given by Definition 1, at least to first order. In practice this means that that calendar and holiday effects can be safely removed from the transformed data prior to forming the required components given by (19), (21) or (23).

Finally, since the procedure advocated is to decompose the original data into an additive decomposition, the aggregated model given by (8) can be fitted by successive application of the techniques given here, first to the individual series $Y_t^{(j)}$ and then to the aggregate series $Y_t$.

## 4 Numerical studies

In keeping with Thomson and Ozaki (1992) the analysis and simulations undertaken in this section are based on a selection of New Zealand official series over the 12 year period 1980 - 1991. The series considered are short-term visitor arrivals, merchandise trade exports and merchandise trade imports.

We first consider the trends and seasonally adjusted series obtained from the New Zealand Visitor arrivals data using (1) with $p = 0$ and $p = 0.5$. The former is the more natural transformation although arguments can be advanced for both; indeed the power transformation chosen by SABL was (3) with $p = 0.25$, a compromise between the two alternatives. The effects of the corrections given in Section 4 are illustrated in Table 1 and Figure 2.

The mean trend bias referred to in Table 1 gives the mean of the differences between the X-11 trend and each of the trends given by $\phi^{-1}(T_t)$, (19), (21) and (23). Since X-11 fits the additive decomposition model (5) directly without any correction, its trend has been used as the basis for comparison. The mean seasonal balance bias measures the mean difference between the centred 12 month (13 point) moving averages of the original and seasonally adjusted series. Here the trend $T_t$ and other components have been estimated

| Adjustment | Mean Trend Bias | | Seasonal Balance | |
|---|---|---|---|---|
| Method | $p = 0$ | $p = 0.5$ | $p = 0$ | $p = 0.5$ |
| No correction | -2040 | -1181 | 1957 | 985 |
| Equation (19) | -47 | -186 | 1 | 1 |
| Equation (21) | -99 | -186 | 53 | -4 |
| Equation (23) | -133 | -181 | 1 | 1 |

Table 1: Visitor arrivals to New Zealand by month; trends and seasonally adjusted series obtained using (1) with logarithm and square root transformations. The mean trend bias (by comparison to X-11) and the mean seasonal balance bias from zero are given for the corrected and uncorrected series.

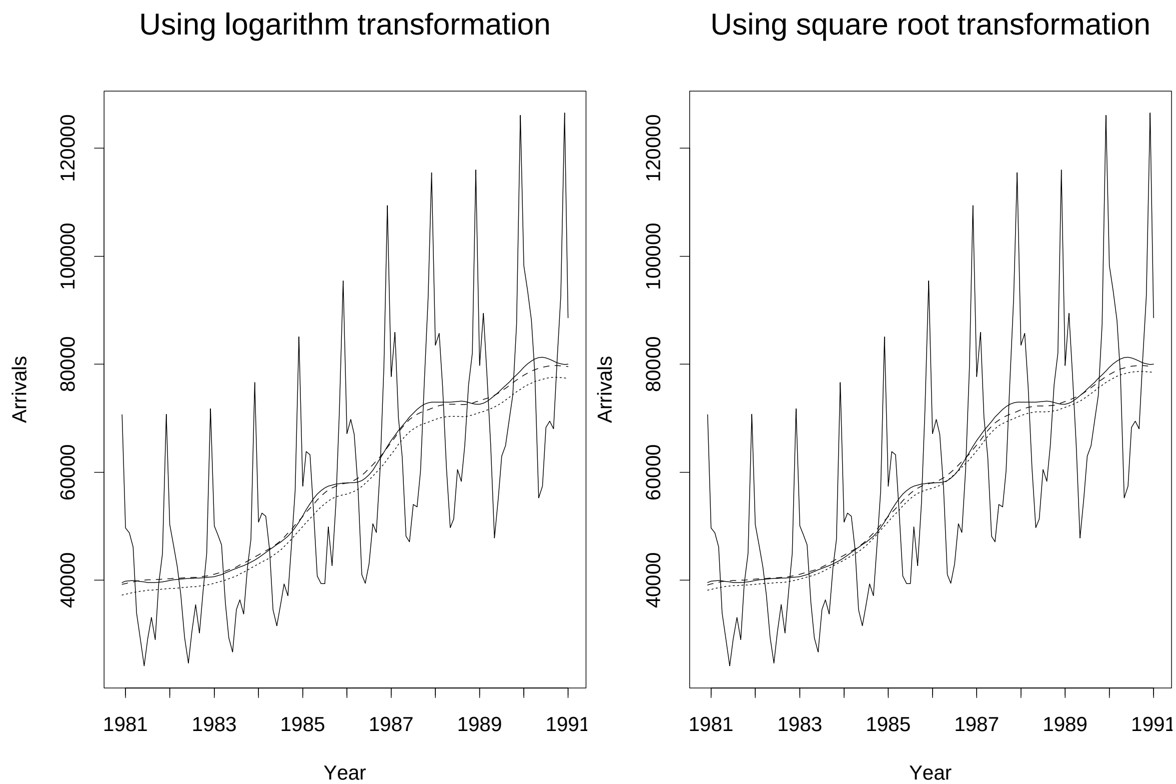



Figure 2: Visitor arrivals to New Zealand by month; trends obtained using (1) with logarithm and square root transformations. In each case three trends are superimposed: the uncorrected trend (dotted line), the corrected trend using (19) (dashed line) and the X-11 trend (solid line).

from the transformed series $\phi(Y_t)$ using SABL. All measurements are in the original scale of the observations and the calculations have been carried out for the various corrected and uncorrected series over the central 10 year period to avoid complications with filter end effects. Figure 2 plots the various trends with and without correction.

The results in Table 1 and Figure 2 indicate that, in the case of strong seasonality, the corrections are a marked improvement over the usual procedure of no correction. There is little to pick between the direct adjustment (19) and its approximations. As might be expected, the unadjusted trend obtained using the square root transformation is better than that obtained using the logarithm transformation. Moreover the corrected trend using the logarithm transformation appears to be better than the corrected trend using the square root transformation. However both corrected trends approximate the X-11 trend reasonably well irrespective of the transformation adopted. Thus the correction procedure results in trends that are, to a large extent, invariant with respect to the transformation chosen

We now consider analyses of three different types of simulated series whose key parameters are given in Table 2. For each type of series, 20 independent realisations of 12 years duration were generated using (1) with power $p$ given by Table 2. The trends were deterministic linear or quadratic functions of time, the seasonal components were fixed non-evolutionary annual cycles, and the irregular components were Gaussian white noise. All components were generated for the transformed series which were then transformed back into the original scale of the observations. The model parameters were chosen following an analysis of the actual series concerned. However, hese analyses were used as a guide only and the parameters adopted provide, at best, an overly simplistic description of the series concerned.

| Simulated Series | $p$ | CV | SI |
|---|---|---|---|
| Visitor Arrivals | 0.0 | 0.05 | 5 |
| Exports | 0.5 | 0.10 | 2 |
| Imports | 0.0 | 0.10 | 1 |

Table 2: Key parameters for series simulated using (1) with power $p$, coefficient of variation CV and seasonal to irregular ratio SI.

The key parameters given in Table 2 are CV, the average coefficient of variation in the original scale of the observations, and SI, the seasonal to irregular ratio $\text{rms}(S_t)/\sigma$ in the transformed scale. Here $\sigma^2$ is the (constant) variance of the irregular component and $\text{rms}(S_t)$ is the root mean square of the seasonal pattern over any 12 month period. Simulated exports and imports have a relatively high variability about the mean level of approximately 10%. Moreover, simulated exports have seasonal amplitudes approximately twice the size of the irregular whereas simulated imports have seasonal amplitudes of approximately the same size as the irregular. Thus, compared to visitor arrivals, simulated exports and imports represent situations where the use of correction formulae should be more marginal.

In Figure 3 examples of simulated visitor arrivals and simulated exports are given together with uncorrected trends, trends corrected using (19) and true trends. Simulated imports present a similar picture to that of exports. In each case the true trend was taken as $T_t^*$ defined by (15) and evaluated for the true $T_t$, $S_t$ and $\sigma$.

The results are summarised in Figure 4. Here the standardised trend bias at a given point in time $t$ is defined as $\Delta\bar{T}_t/(s_{\Delta T}/\sqrt{20})$ where $\Delta\bar{T}_t$ and $s^2_{\Delta T}$ are the sample mean and variance respectively of the differences between the 20 individual trend estimates (with and without correction) and the true trend $T_t^*$. The standardised seasonal balance bias is defined similarly as $\Delta\bar{S}_t/(s_{\Delta S}/\sqrt{20})$ where $\Delta\bar{S}_t$ and $s^2_{\Delta S}$ are the sample mean and variance respectively of the differences between the 20 individual centred 12 month (13 point) moving averages of the original series and their seasonally adjusted forms. Thus the mean trend or seasonal balance bias at any given time point has been measured in units of its own standard deviation.

As before, the results displayed relate to the central 8 year period of the series to avoid possible end effect complications. Only the results for the correction procedure (19) have been displayed in Figure 4 since the other procedures (21) and (23) produce much the same results. The choice of correction procedure can thus be based on other criteria

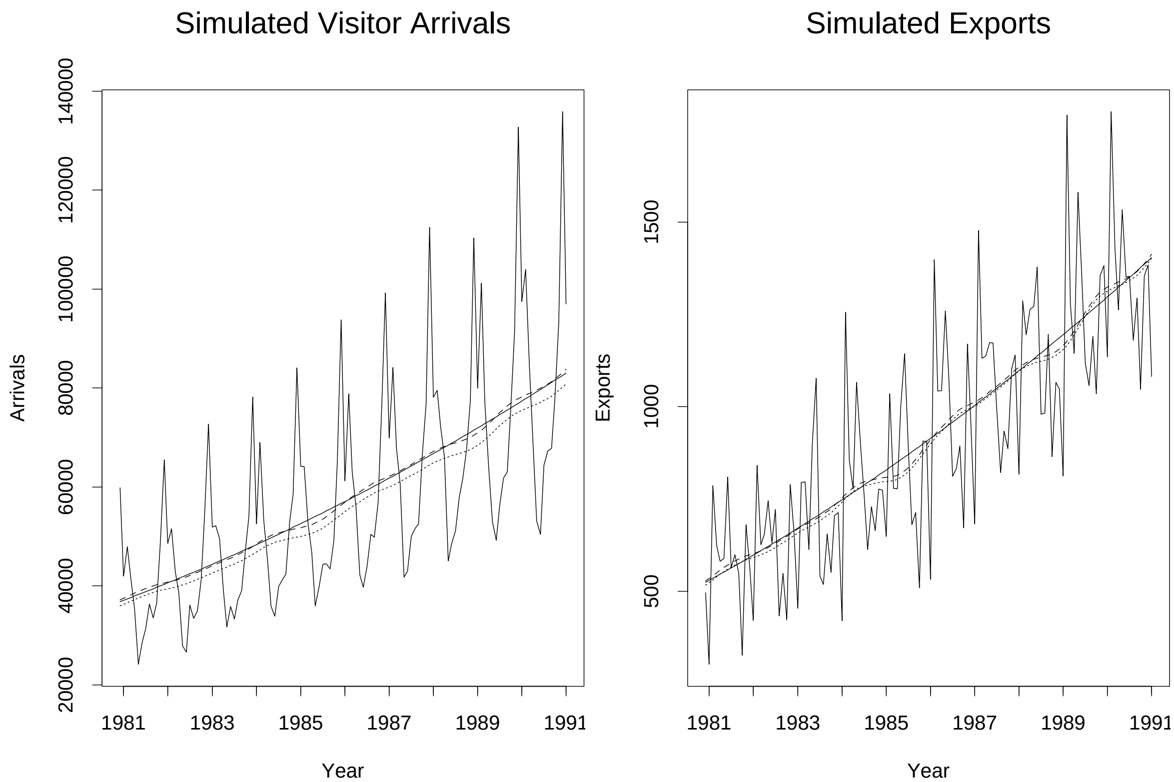
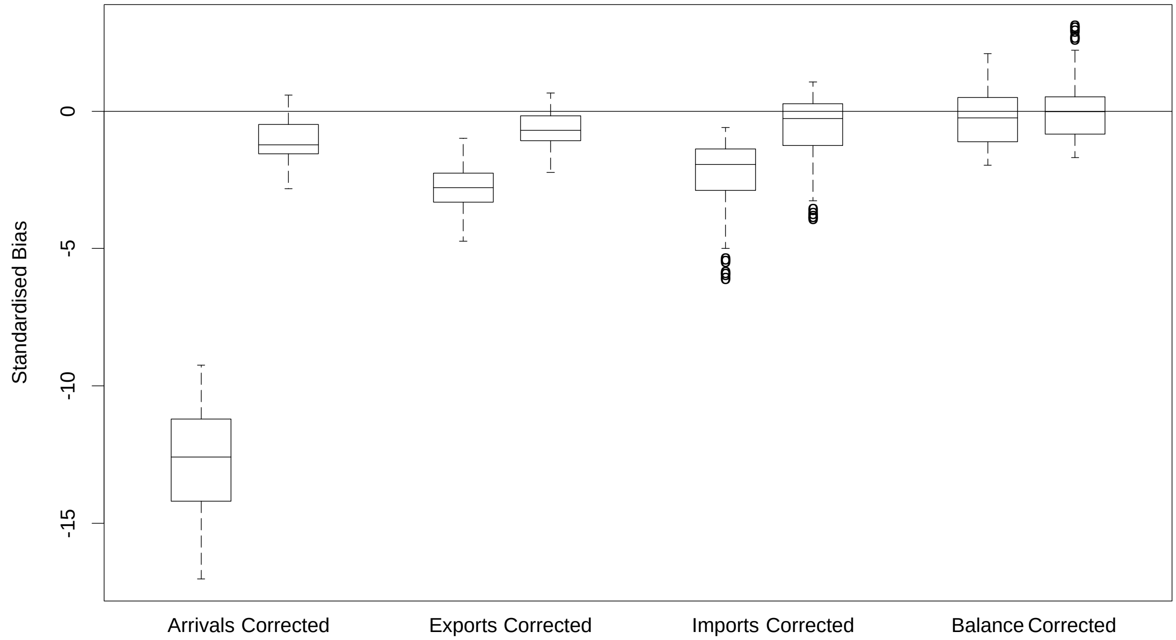



Figure 3: Typical simulations of visitor arrivals and exports to New Zealand by month; trends obtained using (1) with logarithm and square root transformations respectively. In each case three trends are superimposed: the uncorrected trend (dotted line), the corrected trend using (19) (dashed line) and the true trend (solid line).



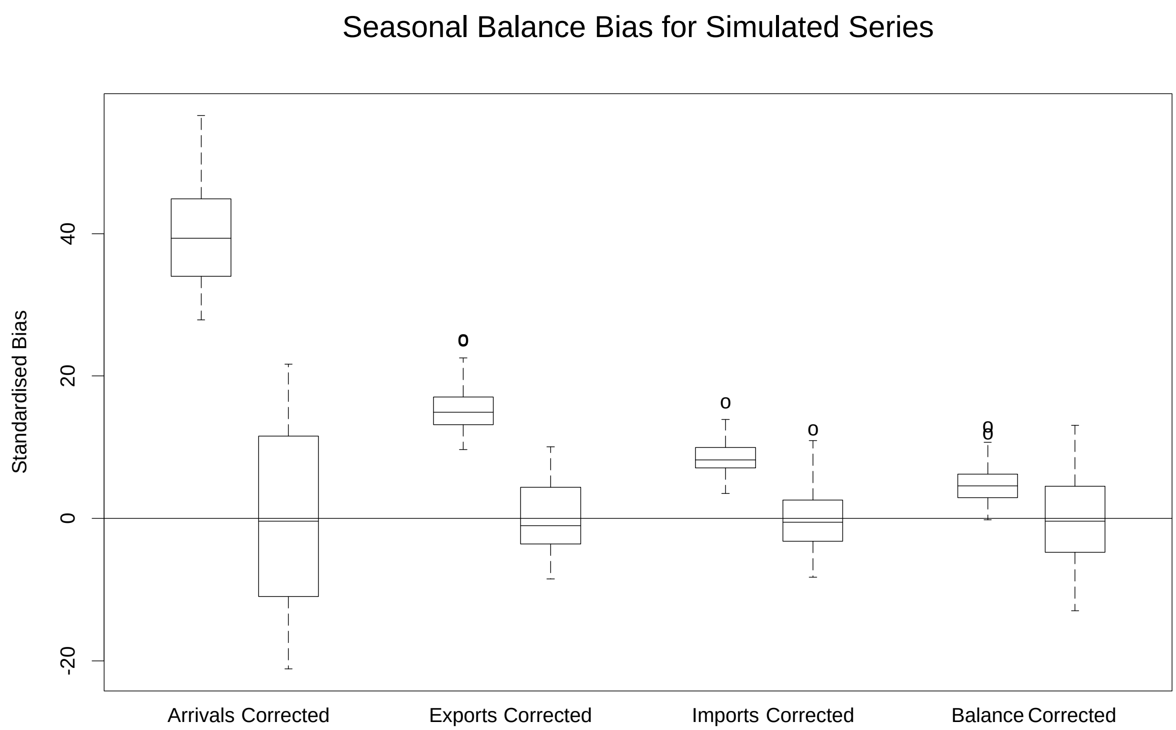




Figure 4: Simulated visitor arrivals, exports, imports and balance of exports less imports; trend and seasonal balance biases. In each case boxplots of standardised trend biases from the true trend and standardised seasonal balance biases from zero are given for the uncorrected series and series corrected using (19).

such as theoretical considerations and computational convenience. Included in Figure 4 is the trend and seasonal balance biases for the balance of exports over imports. Here the components of the simulated exports and imports have been estimated to obtain trends and seasonally adjusted series and then combined. (See the discussion in Section 3.)

The results are self evident; the greater the variation about the trend the greater the gains obtained from using the correction formulae. Even in the case of exports and imports where the seasonal and irregular amplitudes are of modest size, there are still significant gains to be had. These remarks apply to both trend and seasonal balance biases. The reductions in trend bias for the simulated balance series are the least convincing due to the fact that the component series, exports and imports, both have positive trend biases of comparable magnitude which tend to cancel. The trend biases of the balance series would be more marked if the component series had biases that were significantly different as would be the case if the two transformations were the identity and logarithm transformations for example. In the case of biases of the opposite sign, the balance trend biases will exceed those of its component parts.

In the case of the corrected trends, note that there remains a small downwards bias. This is most likely due to the fact that $T_t^*$ has been estimated from (19) with $T_t$, $S_t$ and $\sigma^2$ replaced by estimates from the decomposition of the transformed data. For parametric Gaussian based models this problem might be alleviated to some extent by using (20) and the Kalman filter. For models such as X-11 which are already in the appropriate additive form, no correction formulae are needed and direct fitting of the components should be largely free of trend bias. Procedures that directly fit the X-11 model include X-11-ARIMA, X-12-ARIMA or the parametric procedures of Ozaki and Thomson (1992). However, in general, further corrections will be needed to eliminate estimation bias. This is beyond the scope of the current paper. Finally note that the correction procedures do not appear to increase the variability of the trend estimates; indeed, if anything there might be a slight reduction in variability.

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