Modelling NZ Seismicity with the ETAS Model

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4 May 2011

Abstract

The "Epidemic Type Aftershock Sequence" (ETAS) Model is a point process model used in seismology. It models mainshock-aftershock sequences like an epidemic where an infected individual passes the disease onto others (direct offspring or 1st generation), who in turn pass it on further (2nd and subsequent generations). Given certain conditions, the aftershock sequence, or epidemic, will eventually die out.

We attempt to fit a sequence of spatial ETAS models, from very simple to more complex, to determine the relative importance of the various model components. Each are fitted to a large region that includes all of the most seismically active areas in New Zealand. It is shown that some values of the estimated ETAS parameters, in particular the Omori decay p, depend on the assumed spatial density function of the background process.

We also evaluate the efficacy of the model by examining how well it describes many of the major mainshock-aftershock sequences occurring since 1965 that are contained within the NZ Catalogue.

1 Introduction to Dataset

Data have been extracted from the New Zealand Catalogue

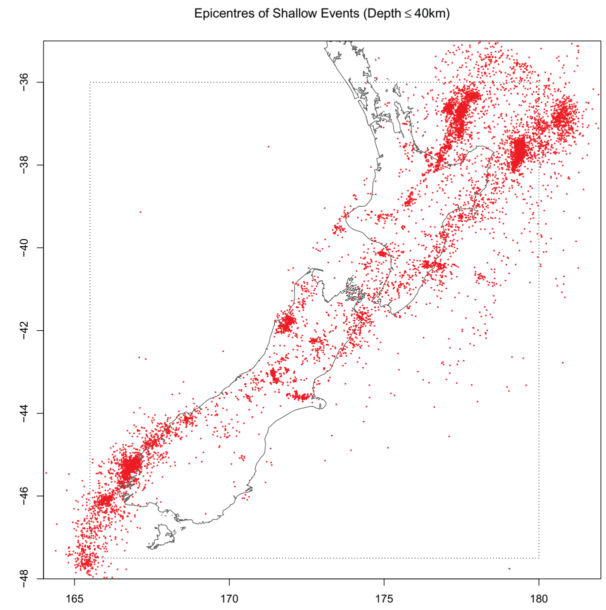
Spatial Boundaries: 164°E – 182°E and 35°S – 48°S

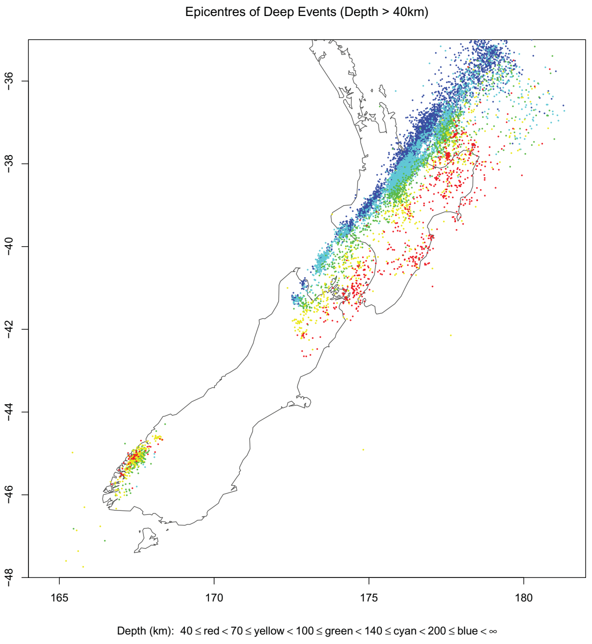
Time Boundaries: 1 January 1965 and 31 December 2010 (inclusive)
i.e. 46 years or 552 months or 16801 days

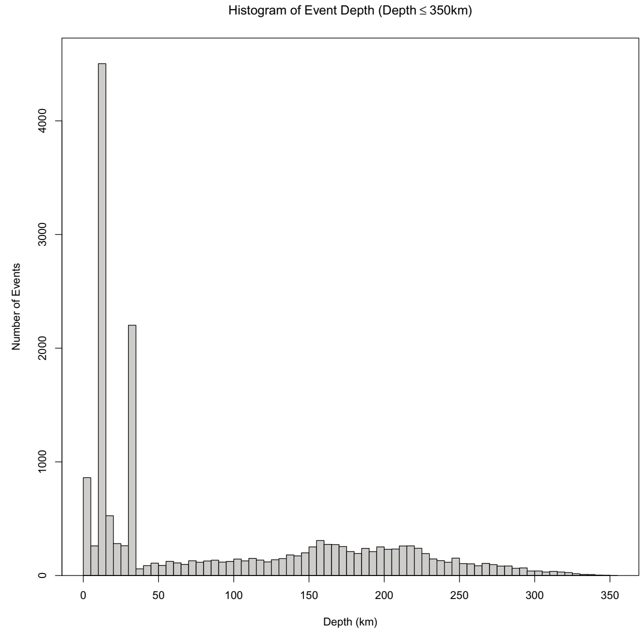
Magnitude Boundary: ≥ 4.0

Total Events: 17379

Initially consider all events regardless of depth



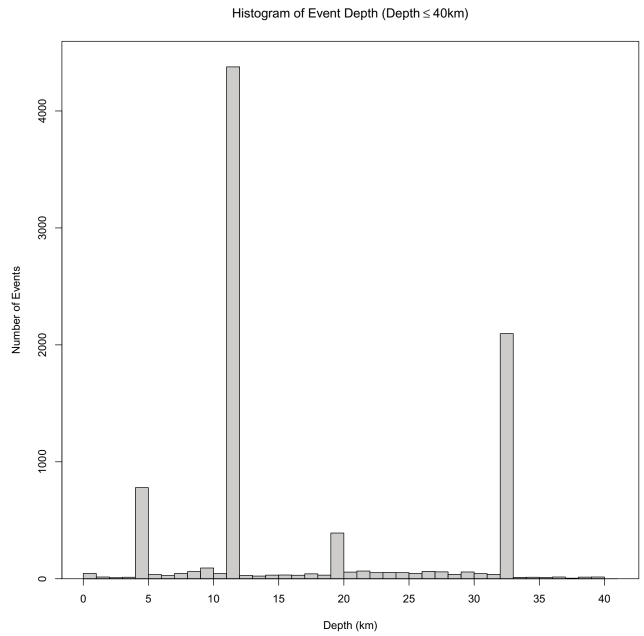


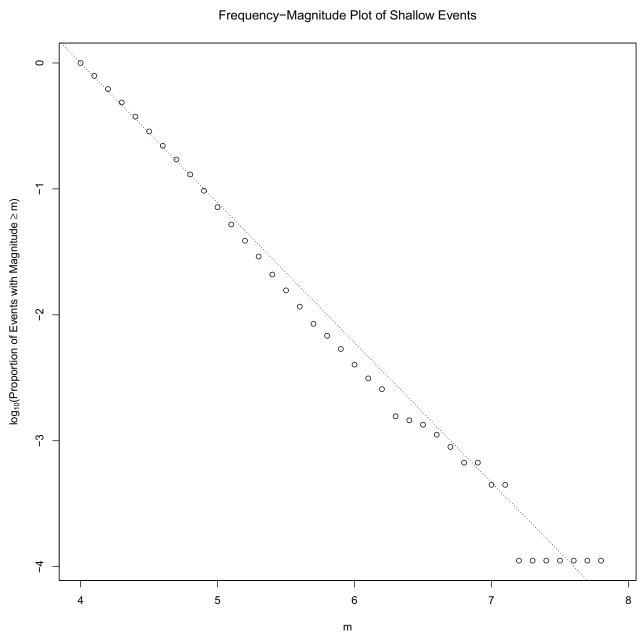


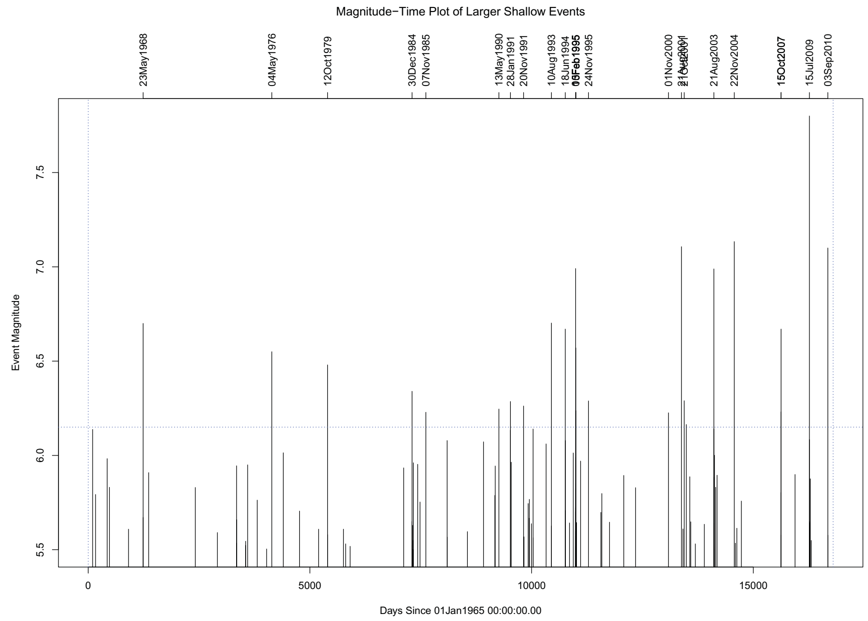
Shallow Events

In subsequent plots and analyses we restrict event depth to ≤ 40km

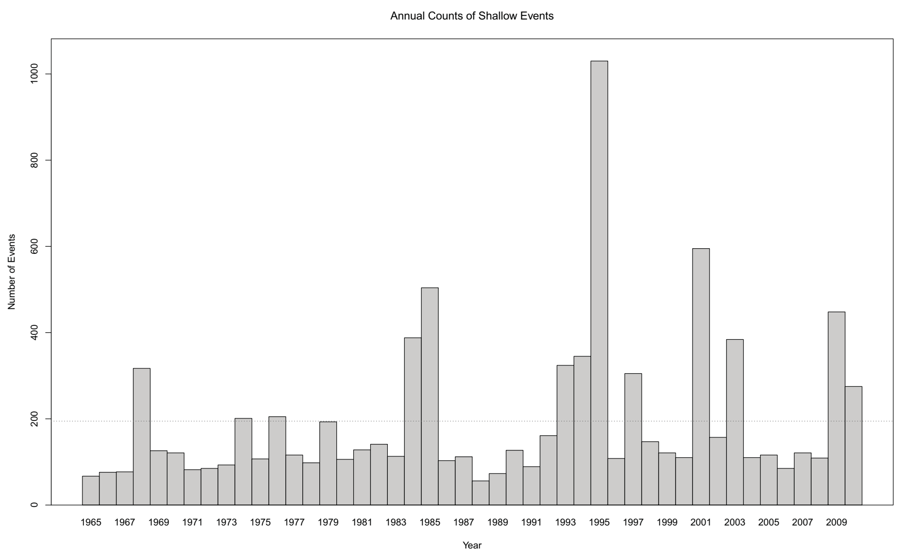
Total Shallow Events: 8954

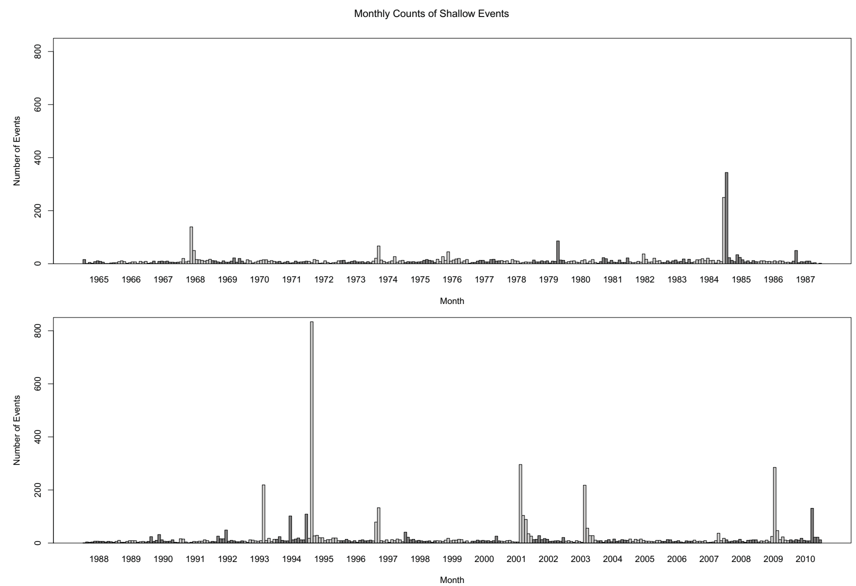


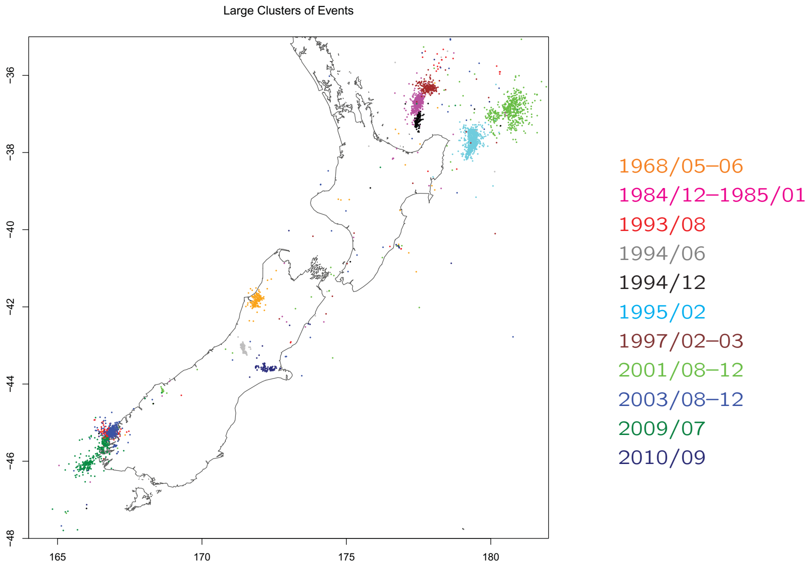




Dates of events with $M \geq 6.2$ are marked on the top axis







Out of the 8954 events in dataset (552 months), there are 3565 events in these 22 months

i.e. 40% of events in 4.2% of time

2 Point Processes

History of process is $\mathcal{H}_t = \{(t_i, w_i, x_i, y_i) \ \forall i : t_i < t\}$

(t_i, w_i, x_i, y_i) is *time, magnitude, longitude, latitude* of event i

Conditional intensity is

$$\lambda(t, w, x, y \,|\, \mathcal{H}_t) = \lim_{\tau,\zeta,\xi,\eta \to 0} \frac{1}{\tau\zeta\xi\eta} \Pr\{N_{\tau\zeta\xi\eta}(t, w, x, y) > 0 \,|\, \mathcal{H}_t\}$$

where $N_{\tau\zeta\xi\eta}(t, w, x, y)$ is number of events in

$$[t, t+\tau) \times [w, w+\zeta) \times [x, x+\xi) \times [y, y+\eta)$$

Marked Point Process

Satisfies

$$\lambda(t, w, x, y \,|\, \mathcal{H}_t) = \lambda_g(t \,|\, \mathcal{H}_t) f(w, x, y)$$

where $\lambda_g(t \,|\, \mathcal{H}_t)$ is the *ground intensity function*
and $f(w, x, y)$ is the *mark density function*

Log Likelihood Function of Marked Point Process

Space over which function is maximised is $\mathcal{T} \times \mathcal{W} \times \mathcal{X} \times \mathcal{Y}$

Let $\mathcal{I} = \{i : (t_i, w_i, x_i, y_i) \in \mathcal{T} \times \mathcal{W} \times \mathcal{X} \times \mathcal{Y}\}$

$$\begin{aligned}\log L &= \sum_{i \in \mathcal{I}} \log \lambda(t_i, w_i, x_i, y_i \,|\, \mathcal{H}_{t_i}) - \int_{\mathcal{T}} \int_{\mathcal{W}} \int_{\mathcal{Y}} \int_{\mathcal{X}} \lambda(t, w, x, y \,|\, \mathcal{H}_t)\, dw\, dy\, dx\, dt \\ &= \sum_{i \in \mathcal{I}} \log \lambda_g(t_i \,|\, \mathcal{H}_{t_i}) - \int_{\mathcal{T}} \lambda_g(t \,|\, \mathcal{H}_t)\, dt \int_{\mathcal{W}} \int_{\mathcal{Y}} \int_{\mathcal{X}} f(w, x, y)\, dw\, dx\, dy \\ &\qquad + \sum_{i \in \mathcal{I}} \log f(w_i, x_i, y_i)\end{aligned}$$

3 Form of ETAS Models

$$\lambda(t, w, x, y \,|\, \mathcal{H}_t) = \mu(t, w, x, y \,|\, \mathcal{H}_t) + \xi(t, w, x, y \,|\, \mathcal{H}_t)$$

where

$$\xi(t, w, x, y \,|\, \mathcal{H}_t) = \sum_{i : t_i < t} \xi_i(t)\, g_i(w, x, y)$$

$$\xi_i(t) = A e^{\alpha(w_i - \widetilde{w})} \left(1 + \frac{t - t_i}{c}\right)^{-p}$$

$\mu(t, w, x, y \,|\, \mathcal{H}_t)$ is a conditional intensity function describing the "background" events

$g_i(w, x, y)$ is a spatial-magnitude mark distribution of the offspring associated with the ith event

$\xi_i(t)$ can be thought of as a ground intensity function describing temporal behaviour of the aftershock sequence associated with the ith event

Expected Aftershocks

Expected number of "direct" aftershocks associated with ith event (not restricting count to observation region) exists if $p > 1$:

$$\int_{t_i}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{\widetilde{w}}^{\infty} \xi_i(t)\, g_i(w,x,y)\, dw\, dx\, dy\, dt \;=\; \int_{t_i}^{\infty} \xi_i(t)\, dt \;=\; \kappa\, e^{\alpha(w_i-\widetilde{w})}$$

where

$$\kappa = \frac{Ac}{p-1}$$

κ is the expected number of "direct" aftershocks (i.e. direct offspring) of an event with magnitude $\widetilde{w}$

Independence Conditions

$$\lambda(t,w,x,y\,|\,\mathcal{H}_t) = \mu(t,w,x,y\,|\,\mathcal{H}_t) + \sum_{i:t_i<t} \xi_i(t)\, g_i(w,x,y)$$

$$\mu(t,w,x,y\,|\,\mathcal{H}_t) = \mu_g(t\,|\,\mathcal{H}_t)\, f(w,x,y)$$

$$f(w,x,y) = f(x,y)\, f(w)$$

$$g_i(w,x,y) = g_i(x,y)\, g(w) \quad \forall i$$

Generally assumed that $f(w) = g(w)$

Uniform Spatial Density of 'Immigrants'

$$\mu(t,w,x,y\,|\,\mathcal{H}_t) = \mu_g(t\,|\,\mathcal{H}_t)\, f(x,y)\, f(w)$$

$$\mu_g(t\,|\,\mathcal{H}_t) = \mu$$

where μ is the constant daily event rate in $\mathcal{X}\times\mathcal{Y}$

$f(x,y)$ is a uniform distribution such that

$$\int_{\mathcal{Y}}\int_{\mathcal{X}} f(x,y)\, dx\, dy = 1$$

Bivariate Normal Spatial Density of Aftershocks

$g_i(x,y)$ is bivariate normal

$$g_i(x,y) = \frac{1}{2\pi\zeta_i^2}\exp\left(\frac{(x-x_i)^2}{-2\zeta_i^2} + \frac{(y-y_i)^2}{-2\zeta_i^2}\right)$$

or approximately isotropic

$$g_i(x,y) = \frac{\cos(y_i)}{2\pi\zeta_i^2}\exp\left(\frac{(x-x_i)^2}{-2\zeta_i^2/\cos^2(y_i)} + \frac{(y-y_i)^2}{-2\zeta_i^2}\right)$$

where $\zeta_i = \delta e^{\beta(w_i-\widetilde{w})}$

Further

$$\int_{\mathcal{Y}}\int_{\mathcal{X}} g_i(x,y)\, dx\, dy < 1$$

4 Initial Basic Model

Model Characteristics:

- Uniform spatial density for immigrant process
- Isotropic spatial density for aftershocks
- Boundary (space-time) correction
  (likelihood & observation regions marked on epicentral plot of shallow events)

This is referred to as 'Model 4'

Movie of Conditional Intensity

Say we sample $\lambda(t, x, y \,|\, \mathcal{H}_t)$ at regular time points, say, 20 day intervals

A larger event occurring just after the sampling time may have its associated Omori function considerably reduced by the next sampling point

Such an event would not be well represented, Problem!

Hence, divide into small space-time windows of 20 days & $0.2^\circ \times 0.2^\circ$

For each space-time cell, calculate $\int\int\int \lambda(t, x, y \,|\, \mathcal{H}_t)\, dx\, dy\, dt$
(expected number of events in each cell)

For each time interval, plot spatial expectations (into a jpeg file)

Bind jpeg's to be played at about 5 frames per second

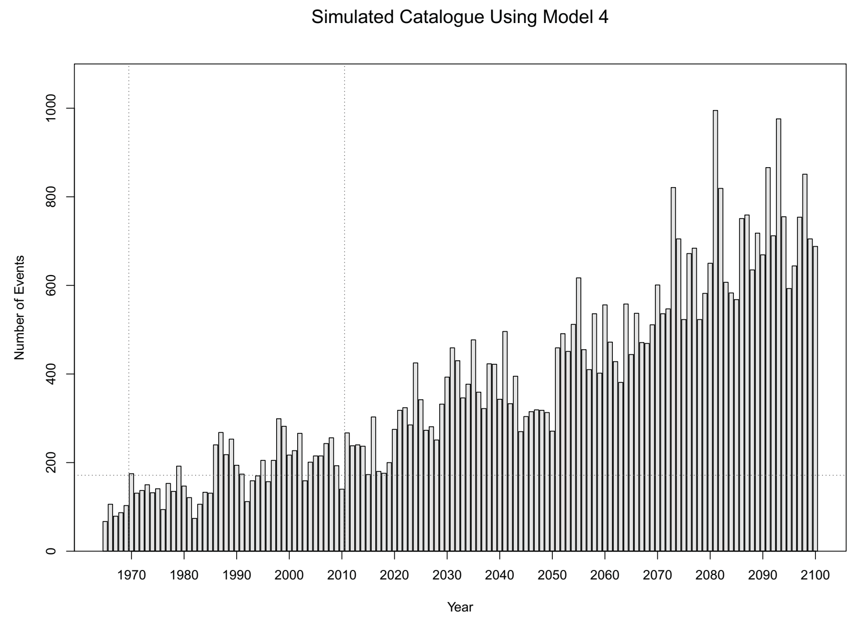
Termination of Aftershock Sequences

If $p < 1$ the expected number of direct aftershocks does not exist
i.e. aftershock sequences do not terminate!

Movie of the conditional intensity function using Model 4 and real data looks quite good
Model is not as good as it looks!

See movie of conditional intensity function using Model 4 and simulated data
Note: increasing seismicity over time and uniform distribution of immigrant (mainshocks or ancestor) events

See annual totals of simulated events until 2100 using Model 4



5 Simple Background Spatial Distribution

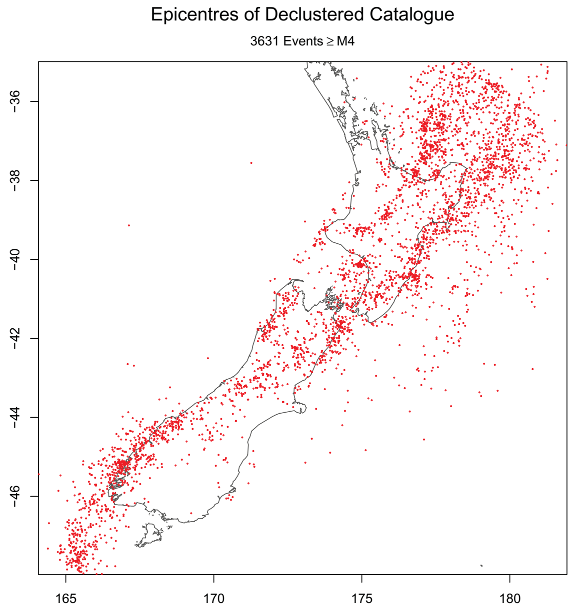
Decluster NZ catalogue between 1965 and 2010

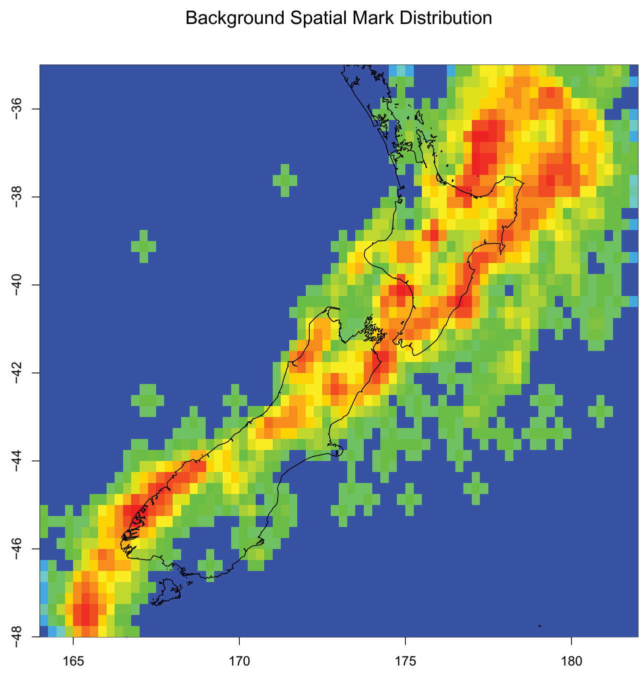
Count binned events in cells of $0.25^\circ \times 0.25^\circ$

Density in each cell is assumed constant

Discontinuous at cell boundaries

Scale so that integral over (164°E, 182°E) × (35°S, 48°S) is one





6 Models with Smooth Spatial Background

Place an isotropic bivariate Gaussian kernel at each of the N event locations (x_i, y_i); $i = 1, \cdots, N$; in the declustered catalogue

Background spatial density is

$$f(x, y; \sigma) = \frac{1}{\Delta_\sigma} \sum_{i=1}^{N} \exp\left\{ \frac{-1}{2\sigma^2} \left[(x - x_i)^2 \cos^2(y_i) + (y - y_i)^2 \right] \right\}$$

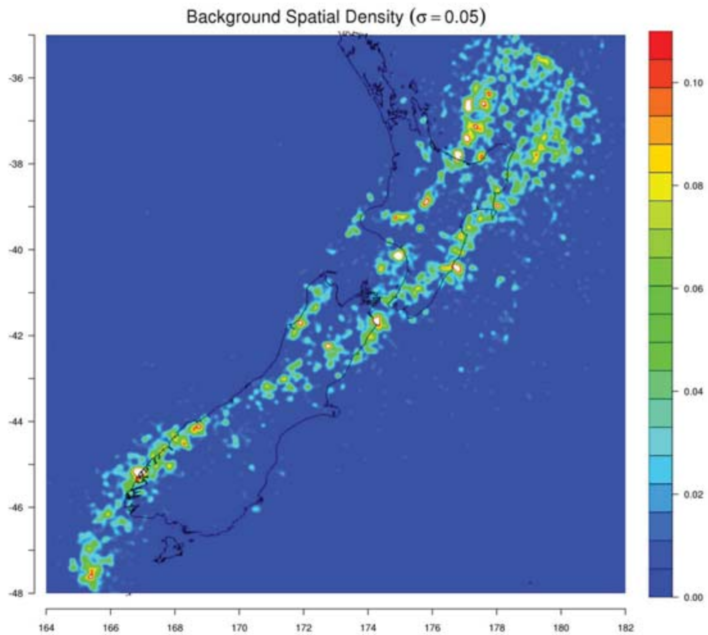
Δ_σ is determined such that

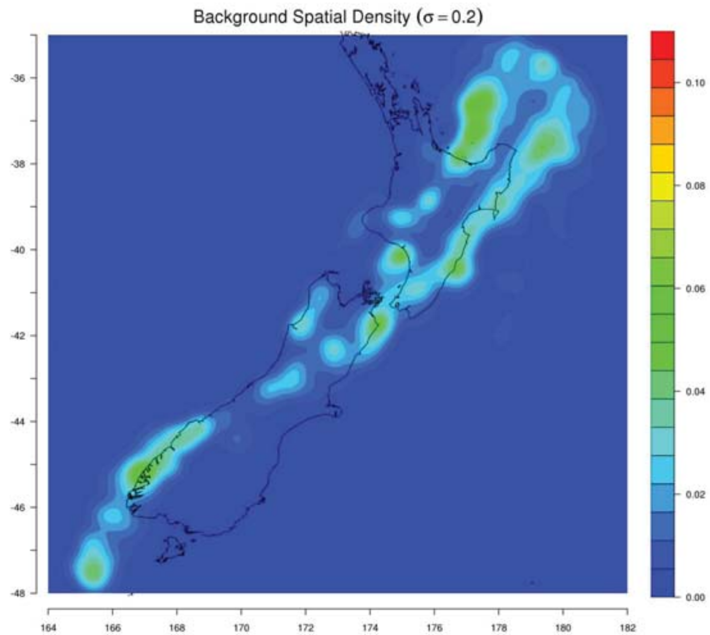
$$\int_{\mathcal{Y}} \int_{\mathcal{X}} f(x, y)\, dx\, dy = 1$$

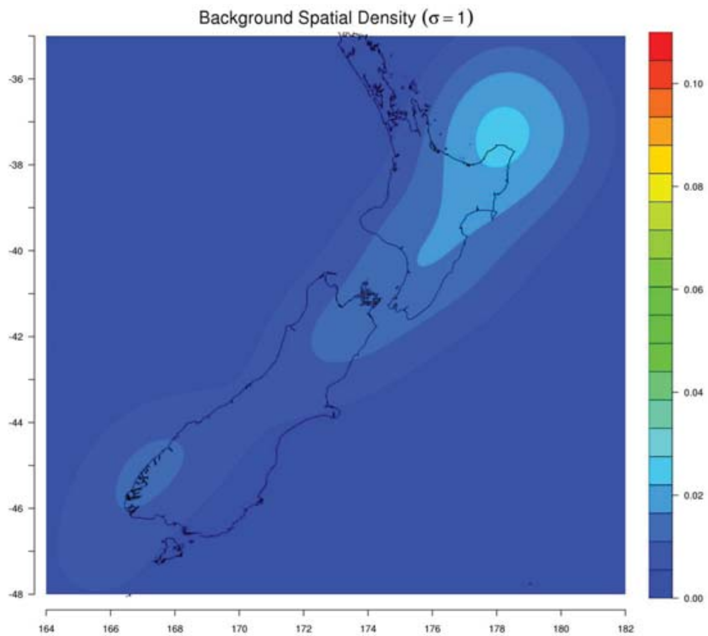
where

$$\mathcal{X} \times \mathcal{Y} = (164^\circ\text{E}, 182^\circ\text{E}) \times (35^\circ\text{S}, 48^\circ\text{S})$$

σ is effectively a smoothing parameter







Effect of Smoothing Background Spatial Density

Referred to as ‘Model 6’ (same boundary correction as in ‘Model 4’)

sigma	Pct	Kappa	mu	A	alpha	c	p	delta	beta	log L1	log L2	log likelihood
0.002	38.08	0.267	0.2129	6.00	1.569	0.0097	1.2190	0.0442	0.4480	7794.5	-427.1	7367.4
0.005	38.23	0.266	0.2137	5.88	1.563	0.0102	1.2268	0.0432	0.4579	3156.6	-427.1	2729.5
0.010	38.66	0.263	0.2162	5.66	1.562	0.0110	1.2375	0.0414	0.4692	-52.8	-427.1	-479.9
0.020	38.96	0.260	0.2179	5.39	1.565	0.0118	1.2445	0.0394	0.4699	-2835.7	-427.1	-3262.8
0.050	38.55	0.262	0.2157	5.23	1.570	0.0117	1.2337	0.0360	0.4810	-5310.7	-427.1	-5737.8
0.100	36.88	0.279	0.2062	5.65	1.555	0.0101	1.2039	0.0356	0.4842	-6319.7	-427.1	-6746.8
0.200	34.63	0.305	0.1936	6.28	1.534	0.0083	1.1708	0.0360	0.4808	-6887.5	-427.1	-7314.6
0.500	30.48	0.370	0.1707	7.53	1.494	0.0060	1.1229	0.0377	0.4687	-7419.9	-427.1	-7847.0
1.000	26.49	0.466	0.1503	8.84	1.454	0.0046	1.0876	0.0399	0.4544	-7791.0	-427.1	-8218.1
2.000	20.74	0.749	0.1203	11.00	1.388	0.0033	1.0482	0.0441	0.4244	-8247.8	-427.1	-8674.9
5.000	13.15	3.721	0.0802	14.25	1.290	0.0022	1.0085	0.0513	0.3697	-8758.9	-427.1	-9186.1
10.000	11.01		0.0704	15.44	1.236	0.0020	0.9996	0.0596	0.2110	-8874.0	-427.1	-9301.1
uniform	10.48		0.0690	15.73	1.226	0.0020	0.9974	0.0608	0.1949	-8905.4	-427.1	-9332.5
0.002	38.08	0.259	0.2128	4.28	1.586	0.015	1.2482	0.0442	0.4455	7779.4	-427.1	7352.3
0.005	38.26	0.259	0.2138	4.38	1.579	0.015	1.2535	0.0431	0.4562	3144.9	-427.1	2717.8
0.010	38.74	0.257	0.2166	4.47	1.574	0.015	1.2606	0.0413	0.4684	-60.4	-427.1	-487.5
0.020	39.10	0.255	0.2187	4.49	1.575	0.015	1.2640	0.0392	0.4700	-2840.2	-427.1	-3267.3
0.050	38.80	0.256	0.2171	4.35	1.580	0.015	1.2544	0.0357	0.4824	-5315.2	-427.1	-5742.3
0.100	37.37	0.267	0.2090	4.19	1.572	0.015	1.2352	0.0352	0.4865	-6331.1	-427.1	-6758.2
0.200	35.49	0.283	0.1984	4.02	1.560	0.015	1.2133	0.0354	0.4844	-6912.0	-427.1	-7339.1
0.500	31.99	0.316	0.1792	3.77	1.537	0.015	1.1786	0.0366	0.4749	-7477.0	-427.1	-7904.1
1.000	28.56	0.357	0.1621	3.57	1.513	0.015	1.1501	0.0382	0.4640	-7885.4	-427.1	-8312.5
2.000	23.27	0.444	0.1350	3.37	1.465	0.015	1.1139	0.0420	0.4336	-8403.1	-427.1	-8830.2
5.000	15.45	0.670	0.0942	3.18	1.380	0.015	1.0712	0.0494	0.3703	-8999.6	-427.1	-9426.7
10.000	13.36	0.769	0.0854	3.15	1.355	0.015	1.0614	0.0522	0.3426	-9135.1	-427.1	-9562.2
uniform	12.78	0.802	0.0842	3.14	1.347	0.015	1.0587	0.0530	0.3326	-9173.4	-427.1	-9600.5

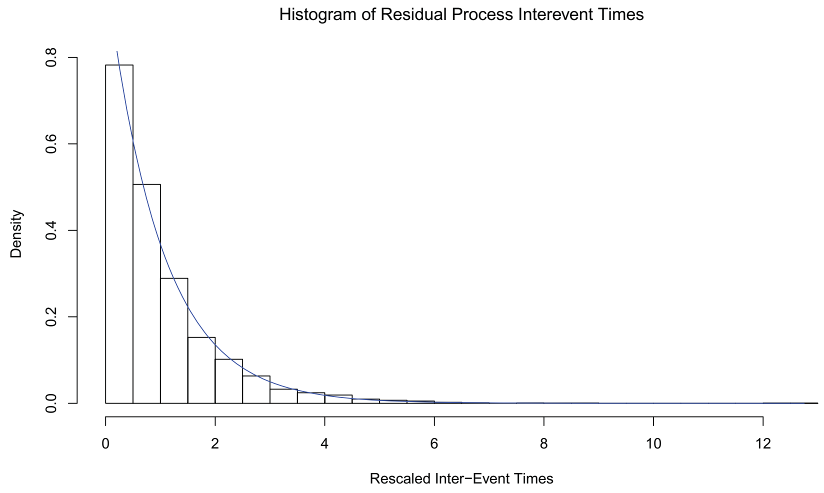
7 Goodness of Fit

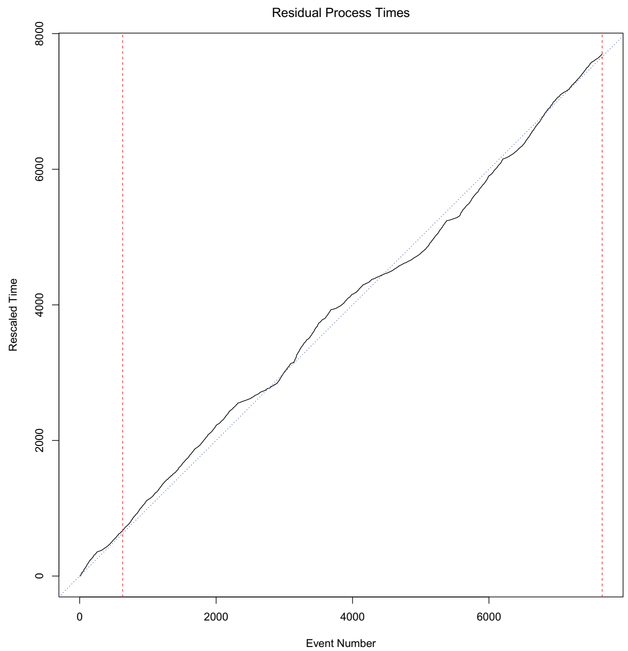
Following analyses are on ‘Model 6’ with $\sigma = 0.2$ and $c = 0.0083$

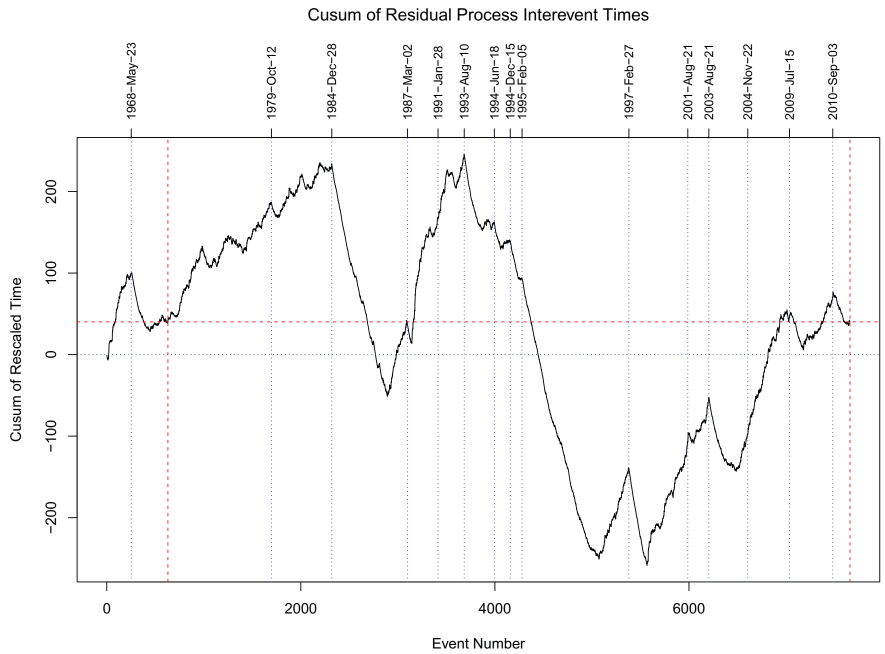
Residual Process

$$\tau_i = \int_0^{t_i} \int_{\mathcal{Y}} \int_{\mathcal{X}} \int_{\mathcal{M}} \lambda(t, x, y, M \mid \mathcal{H}_t)\, dM\, dx\, dy\, dt$$

If the events have been generated by the assumed model, then $\{\tau_i : i = 1, 2, \cdots\}$ represents event times of a Poisson process with rate one







Examples of Large Sequences

Event Sequence	Start Time		X1	X2	Y1	Y2	Max Mag	N	E[N]	N/E[N]
Hawks Crag Westport	1991-01-28	12:58:48	171.3	172.0	-42.3	-41.6	6.3	16	24.01	0.67
Dusky Sound	2009-07-15	09:22:29	165.5	167.4	-46.7	-44.8	7.8	317	302.08	1.05
Milford Sound	1976-05-04	13:56:29	167.0	168.0	-45.0	-44.3	6.6	28	25.11	1.12
Darfield	2010-09-03	16:35:41	171.5	173.3	-44.0	-43.0	7.1	134	97.82	1.37
Puysegur Bank	1979-10-12	10:25:22	165.5	167.0	-47.5	-45.8	6.5	66	44.99	1.47
Arthurs Pass	1994-06-18	03:25:15	171.0	171.8	-43.4	-42.6	6.7	99	67.39	1.47
Weber II	1990-05-13	04:23:10	176.0	176.8	-40.6	-40.1	6.2	29	19.38	1.50
Fiordland	2003-08-21	12:12:50	166.3	167.6	-45.9	-44.7	7.0	277	183.37	1.51
Edge0umbe	1987-03-02	01:42:35	176.5	177.2	-38.3	-37.5	6.1	45	29.13	1.54
Offshore East Cape	1995-02-05	22:51:02	178.0	180.5	-38.5	-37.0	7.0	851	526.87	1.62
Inangahua	1968-05-23	17:24:17	171.4	172.5	-42.4	-41.2	6.7	179	108.13	1.66
Se0retary Island	1993-08-10	00:51:52	166.3	167.4	-45.7	-44.9	6.7	195	108.58	1.80
Bay of Plenty	1994-12-15	11:20:20	176.8	178.0	-37.8	-36.6	6.0	111	59.67	1.86
Bay of Plenty	1984-12-28	00:00:00	176.7	178.2	-37.7	-36.0	6.3	576	307.14	1.88
Bay of Plenty	1997-02-27	00:00:00	177.0	180.0	-37.8	-35.5	5.6	204	102.81	1.98

Where Are the Other “Expected” Events?

Tasman Sea: $(164°\text{E}, 170°\text{E}) \times (35°\text{S}, 42°\text{S})$, 1975–2010

$N = 1$, $\text{E}[N] = 1.36$

In areas with active seismicity with no significant aftershock sequences

Consider Central NZ: $(172°\text{E}, 177°\text{E}) \times (40°\text{S}, 42.5°\text{S})$
$M \geq 4$, depth $\leq$ 40km:

Year	N	E[N]
2006	20	25.23
2007	15	22.04
2008	14	24.26
2009	11	21.21
2010	19	26.75

8 Summary of Model Deficiencies

Under fitting major mainshock-aftershock sequences

Over fitting in space-time regions of ‘normal’ seismicity

Boundary problem in time, space (including depth), and magnitude; e.g. excludes Gisborne (20 Dec 2007)

Can orientation of $g_i(x, y)$ be determined from the main shock (eg. moment tensors)

Spatial variation of p related to crustal properties

Magnitude being independent of space

Powerlaw tail of ETAS spatial mark distribution

Further analyses and examples can be found at

http://homepages.maxnet.co.nz/davidharte/SSLib

under ‘Examples’