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Regeneration and Markovianity

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David:
Peter Whittle, David Kendall, Pat Moran

Daryl:
Ron Bainbridge, Peter Finch, David Kendall, Pat Moran

PROBABILITY: (1) Independence, (2) Expectations

BOOKS: Euclid
Newton–Leibniz (limits, esp. calculus)
Kolmogorov (countable infinity of events)

Feller *Introduction to Probability and its Applications*, 1950
Second edition = Volume 1 of 2-volume work, 1957
Third Edition, 1968

Volume 2, 1966 (second edition, 1971).

'Recurrent events', now called regenerative phenomena
(regeneration points in Palm's work, DGK emphasis 1951)
Renewal sequence (esp. mid 60s, DGK): $\{u_n\}$
Linked renewal sequences:
Discrete-time Markov chains, $\{p_{ij}^n\}$, limit properties

In November 2011 Phil Pollett emphasized

Theorem (DV-J, late 50s or early 60s). *For all states i, j of an irreducible Markov chain, the power series $\sum_n p_{ij}^{(n)} z^n$ have* (a) *the same radius of convergence, $|z| = R$ say, and* (b) *the same cgce behaviour for any given z on the circle of cgce.*

('Solidarity' properties).

DGK's embedded Markov chain analysis (1951, 1953).
Notably for queueing systems, other applications
Discrete- and continuous-time renewal theory (Takacs)

Blackwell's renewal theorem, Smith's Key Renewal Thm
Coupling proofs mid 70s (Pitman, Lindvall; Thorisson)

John Kingman *Regenerative Phenomena* Wiley, c.1972

1964, *Z. Wahrs.* Reg Phen, in cts time p-functions
countable state space Markov chains in cts. time
birth–death processes

Interpretation: $Z(t)$ is a cts time $\{0,1\}$-valued process.
$Z(t) = 0$ means that Z is continuously regenerating itself, rate q determines exponentially distributed duration.
Time-intervals where $Z(t) = 1$ are i.i.d., d.f. F say.

Z is alternating renewal process.

Principal aim:
Characterize diagonal transition functions $p_{ij}(\cdot)$.

Regeneration: If $\{u'_n\}$ and $\{u''_n\}$ are renewal sequences so is $\{u'_n u''_n\}$.

If p_1, p_2 are cts time p-functions, so is $p_1 p_2$.

If h_1, h_2 are renewal density functions, is $h_1 h_2$?

True if h_1 = constant ≤ 1 (probability proof: geometric sampling)
False in general: If $\alpha\, h$ is renewal density for all $\alpha > 0$, then h is effectively a p-function. [Lifetime d.f. is from alternating renewal process with one component exponential.]

BUT: If h_1, h_2 are such renewal density functions, then so is $h_1 h_2$ (because they are effectively p-functions, and p-functions are closed under products).

HOWEVER, product formation can change from 'recurrent' function to 'transient' function

PROBLEM: Given renewal process with generic lifetime d.f. F and renewal function $H = \sum_{n=1}^{\infty} F^{n*}$,
find $\alpha_F := \sup\{\alpha : \alpha H \text{ is a renewal function}\}$.

Limit properties in MCs and point process problems proved via sub- or super-additivity:

$$\text{(sub):} \qquad g(x+y) \le g(x) + g(y)$$

$\lambda := \inf_{x>0} g(x)/x$ exists finite or $-\infty$ and $\lim_{x\to\infty} \dfrac{g(x)}{x} = \lambda$.

$$\text{(super):} \qquad h(x+y) \ge h(x) + h(y)$$

$\mu := \sup_{x>0} h(x)/x$ exists finite or $+\infty$ and $\lim_{x\to\infty} \dfrac{h(x)}{x} = \mu$.

Renewal function

$U(x) = \sum_{n=0}^{\infty} F^{n*}(x) = \mathrm{E}\big(N[0,x]\big)$ is subadditive

For a stationary point process N on $\mathbb{R}$, second moment fn
(i): $M_2(x) := \mathrm{E}\big([N(0,x]\,]^2\big)$ is superadditive.
(ii): $\varphi(x) := \Pr\{N(0,x] > 0\}$ is subadditive
yields Khinchin's existence theorem: $\inf_{x>0} \varphi(x)/x$ exists, and (Korolyuk's theorem) equals $\mathrm{E}(N(0,1])$ when N is orderly

$U_p(x) := \int_0^x p(t)\,\mathrm{d}t$ for p-function is subadditive:
$U_p(x+y) \le U_p(x) + U_p(y)$

[JFCK: 'a curious property, use unknown' (!)]

Notice that $U_p(x) = \mathrm{E}(T_x)$ where T_x = time during $(0,x)$ that reg. phen. is in regenerative state conditional on being in reg. state at time 0: $\mathrm{E}(T_{x+y}) \le \mathrm{E}(T_x) + \mathrm{E}(T_y)$.
PROBLEM: For Markov renewal process, on d states, Prove: $U(x) + \frac{1}{2}(d-1)$ is subadditive; $\frac{1}{2}(d-1)$ is best possible constant. [$U(x) =$ E(no. states visited in $[0,x]$ | a state is entered at 0)].

A PROBLEM re embedded point process that is regenerative (meaning: $\{t_n\}$ epochs of pt process, a subset $\{t'_n\}$ of these epochs is a renewal process)

E.g.: Departure process N_{dep} of stationary $M/M/k/K$ system is not renewal, but it is irreducible Markov renewal. As a point process, there is embedded in it a sequence of regenerative epochs:

What can be said about limit properties of a point process containing an embedded regenerative structure?

In particular, variance asymptotics (for $t \to \infty$):

$$\operatorname{var} N(0,t] = A\,t + o(t) \qquad \text{(crude asymptotics)}$$
$$\operatorname{var} N(0,t] = A\,t + B + o(1) \qquad \text{(fine asymptotics)}$$

PROBLEMS:
(i) How to formulate embedded reg. structure ?
(ii) Prove fine asymptotics for N via embedded property?

For a stationary renewal process, the fine asymptotics hold as soon as the lifetime distribution has a third moment.
Do these carry over to a stationary point process that contains an embedded regenerative structure ? [e.g. true for suitable Markov renewal processes].

Possible technique: $\operatorname{var} N(0,t]$ has integral representation (for orderly stationary N):

$$\operatorname{var} N(0,t] = mt + 2\int_0^t [U(u) - mu]\, m\, \mathrm{d}u$$

Now $mt = \mathrm{E}\bigl(N(0,t]\bigr)$ for stationary point process N, so

$$\frac{\operatorname{var} N(0,t]}{\mathrm{E}\bigl(N(0,t]\bigr)} - 1 = \frac{2}{t}\int_0^t [U(u) - mu]\, \mathrm{d}u,$$

so the limit as $t \to \infty$ of LHS depends on rate of convergence of $U(u) - mu$ to its limit (if it exists): for renewal process, limit $= \mathrm{E}(X^2)/2[(E(X))^2]$, (or, absorbing mt)

$$\frac{\operatorname{var} N(0,t]}{t/\mathrm{E}(X)} \to \frac{\operatorname{var} X}{[\mathrm{E}(X)]^2}$$

Renewal thm does not yield full detail of convergence rate

Finite third moment yields finiteness on

$$U(u) - mu - \tfrac{1}{2}(\text{approx'n to 2nd moment}),$$

hence

$$\operatorname{var} N(0,t] = \frac{\operatorname{var} X}{[\mathrm{E}(X)]^2}\, t + \frac{1}{2}\left[\frac{\mathrm{E}(X^2)}{[\mathrm{E}(X)]^2}\right]^2 - \frac{\mathrm{E}(X^3)}{3[\mathrm{E}(X)]^3} + o(1).$$

Analogue of this relation exists for Markov renewal process (MRP) on finite (or countably infinite ?) state space $\mathbb{X} = \{i, j, \ldots\}$, provided first-entrance r.v.s $T_{ij} =$ time for first jump into state j when there has been jump into state i at $t = 0$, have finite moments of first and second orders.

$$\lim_{t\to\infty} \frac{\operatorname{var} N(0,t]}{\mathrm{E}\bigl(N(0,t]\bigr)} = 1 + \sum_{j\in\mathbb{X}} \left[\lambda_j^2 \operatorname{var} T_{jj} + 1 - 2\sum_i \check{\pi}_i \mathrm{E}(T_{ij})\lambda_j\right]$$

where λ_j is rate of occurrence of successive entries to state j,
T_{jj} is generic first-return r.v. to state j,
$\check{\pi}_i = \Pr\{\text{entry at } t \text{ is to state } j \mid N \text{ has jump at } t\}$
The sum $\sum_j \mathrm{E}(T_{ij})\lambda_j$ in the pure Markov chain case is independent of i (proof not obvious; JJH 'Kemeny's constant').
PROBLEM: Does Kemeny's constant extend from MC to MRP case?
[?? Linear algebra problem ? i.e. eigenvector question ? JJH uses generalized inverses for Kemeny's problem for MCs]

CONCLUSION:

Study 'simple' regenerative settings and obtain results for Markov chains: not all problems are solved (!).

[NW08]: Yoni Nazarathy and Gideon Weiss, QUESTA (2008)
BRAVO effect:
Balancing Reduces Asymptotic Variance of Output (of stationary M/M/1/K queue). [Nazarathy in Melbourne at Swinburne Univ Technology: same is true of M/M/k/K, and of M/M/k/Rneg, except (?) change in const $= \frac{2}{3}$ for M/M/1/K].
Have to 'balance' asymptotics of $k, K \to \infty$: $K = \alpha\sqrt{k}$.
These asymptotics are 'correct' for explaining approx. parabola in NW08

$- -> Q(t)$

0 $\qquad\qquad$ k $\qquad\qquad\qquad\qquad$ K

$$\text{`Output'} = \begin{cases} \text{input} & \text{when } \rho < 1, \\ \text{service capacity} & \text{when } \rho > 1, \\ \frac{1}{2}(\text{sum of above}) & \text{when } \rho \approx 1. \end{cases}$$

Phase transition. (Cf. branching process behaviour for mean offspring $<$, $>$ and ≈ 1)
PROBLEM: GI/GI/k/K ?